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M.E VARIATIONS OF THE MULTIPLE-CHOICE QUESTION

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1. Introduction. In attempting to construct multiple-choice items for objective tests in mathematics which will test the students' ability to solve particular kinds of problems certain difficulties are encountered. Thus, for some types of problems, it is possible for the student to determine the correct answer without actually solving the problem when it is stated in the conventional manner. For example, in the problem

Example 1. The roots of the equation $x^2+5x+6=0$ are (1) $-2, -3$ (2) 2, 3 (3) $-6, 1$ (4) none of the proposed answers is correct.*

the correct response can be found by substitution. Consequently, such an item does not necessarily test the ability of the student to solve a quadratic equation. To obviate this possibility some constructors state the problem in such a manner that it is necessary to solve the problem first, and then perform some operation or operations on the answer. Thus, instead of stating the problem in the form given above, it would be given in a manner similar to the following:

Example 2. The sum of the squares of the roots of the equation $x^2+5x+6=0$ is (1) -13 (2) 13 (3) 37 (4) none of the proposed answers is correct.

For problems of this type this is a simple and satisfactory solution of the difficulty. There are other types of problems, however, where such a solution is not possible.

2. Object. It is the purpose of this paper to propose for other types of problems a different variation of the multiple-choice item which the authors have devised in an effort to minimize the possibility of the student determining the correct response without solving the problem. In addition some suggestions will be made on the use of multiple-choice items for testing the students' ability to graph and sketch curves. This is a type of item which has not been used very extensively by other constructors.

The items we wish to discuss formed part of a test which the authors prepared in cooperation with the Division of Educational Reference for the Army Specialized Training Program. Hence the exact problems are restricted. Consequently, the examples which we shall use will not be the items included on this test but they will be similar. The comparison, therefore, cannot be exact. Since

* This fourth response has been used uniformly on the various tests constructed by the authors. If the student is merely guessing this does not change the theoretical probability of guessing the correct response. Since it is often difficult, however, to construct a third plausible distractor it is believed that this procedure is a satisfactory solution in such cases because the student may have devised a better distractor. This scheme permits him to use it. When the fourth response is the correct one then such a scheme permits the student to solve the problem incorrectly yet mark the correct response. It is the opinion of the authors that this weakness is more than compensated for by its advantages.

the problems will follow very closely the problems actually used it is believed that it will be possible to obtain a relatively clear picture of their effectiveness.

The test was given to all students regularly enrolled in AST Mathematics 407. The number taking the test was 385. The validity of each item, that is, how well each item discriminates between students making high grades on the test and those making low grades on the test, was determined. The method used for determining the validity index was that proposed by J. C. Flanagan.* This index is the estimate of the product moment coefficient of correlation between total test score and a given item based on the per cent obtaining the correct response to the item of those scores which were in the upper and lower 27% of the group. Since this is the product moment coefficient of correlation, an index of 0.19 is significant at the 5% level, and an index of 0.25 is significant at the 1% level. The reliability of the test was 0.90 when the short approximation form of the Richardson-Kuder formula was used. It should be pointed out that this formula tends to underestimate so that the actual value will be more than this.

3. A proposed solution. In analytical geometry when it is desired to determine if the student can find the equation of a line through two given points the objective form of this problem commonly used is

Example 3. The equation of the line passing through the points (2, 0) and (1, 2) is

(0.37; 78) (1) $2x-y=4$ (2) $2x+y=4$ (3) $2x-3y+4=0$ (4) none of the proposed answers is correct.

The first number, 0.37, in the parentheses is the validity index and the second number, 78, is the per cent of correct responses. The same notation will be used in the remainder of the examples which will be discussed.

As was mentioned previously the correct response to problems of this kind can be determined by substitution without actually solving the problem even though, in making up the distractors, the incorrect responses have one of the given points as a solution. It should be noted that the validity index of this item was definitely significant. Nevertheless one cannot be sure that the desired objective—to determine whether the student can find the equation of a line through two given points—has been achieved. In trying to devise an item which would require the student to work the problem we included on the test some experimental items. These items directed the student to find the required equation and then identify the determined coefficients with the general form. Thus, in this form Example 3 would be given:

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* J. C. Flanagan, General considerations in the selection of test items and a short method of estimating the product moment coefficient from data at the tails of the distribution. Journal of Educational Psychology 30: 674–680, 1939.

Example 4. The equation of the line through (2, 0) and (1, 2) is

$$ax + by + c = 0 \quad (a \text{ is a positive integer})$$

where

(0.61; 66) $a = (1)\ \ 2\ (2)\ 3\ (3)\ 1\ (4)$ none of the proposed answers is correct,
(0.54; 63) $b = (1)\ -3\ (2)\ 1\ (3)\ 2\ (4)$ none of the proposed answers is correct,
(0.42; 42) $c = (1)\ -4\ (2)\ 4\ (3)\ 6\ (4)$ none of the proposed answers is correct.

For those topics which lend themselves to the use of this method of identification of coefficients it is believed that this type of objective item will encourage the student to work the problem. It certainly makes the trial and error method of substitution for determining the correct answer practically impossible. At the same time it has a tendency to discourage random guessing.

To machine score such a problem each part was scored separately. The correct identification of each coefficient was counted one on the total score.

The validity indexes for this problem indicates the possibility that the validity of the item is increased by this method over that of the same problem when it is given in the conventional form. Further experimentation is being conducted to determine the correctness of these tentative conclusions which were suggested by this pair of parallel problems. At least, for the ten experimental problems of this type included on the test, it can be stated that in every case each response—of which there were 37—was significant at well above the 1% level.

Because the method of determining or stating a correct answer by identification of coefficients is not generally used for solving problems in the classroom it might be thought that this would be confusing to the student, and consequently he would omit these problems. Apparently this was not the case in general since only about 1% omitted problems of this type while items of the conventional type were frequently omitted more often. There is some ambiguity in the statement of problems like Example 4 where the only restriction placed on a is that it be a positive integer. Although this might be confusing to an instructor of mathematics there was no indication during the administration of the test that it caused the student any difficulty. The statistical data also gives no such indication.

4. Some illustrative examples. In order to suggest some of the possible uses of this form of objective problem, and at the same time to give more information about the difficulty and general high validity of such items two additional examples will be given.

Example 5. The equation of the ellipse with semi-major axis of 5 on the x-axis, semi-minor axis of 3, and center at the origin is

$$ax^2 + bx + cy^2 + dy = e \quad (a \text{ is a positive integer})$$

where

(0.57; 66) $a=$ (1) 10 (2) 25 (3) 9 (4) none of the proposed answers is correct,
(0.51; 62) $b=$ (1) 0 (2) -10 (3) -6 (4) none of the proposed answers is correct,
(0.45; 57) $c=$ (1) 6 (2) 25 (3) 9 (4) none of the proposed answers is correct,
(0.72; 74) $d=$ (1) 0 (2) -10 (3) -6 (4) none of the proposed answers is correct,
(0.29; 53) $e=$ (1) 60 (2) 225 (3) 30 (4) none of the proposed answers is correct.

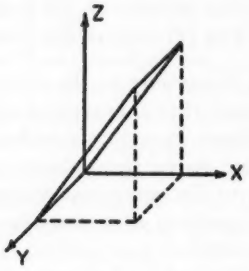


FIG. 1

Example 6. In Fig. 1 the plane PT passes through the point (3, 2, 4). The equation of the plane is

$$Ax + By + Cz + D = 0 \ (A \text{ is a positive integer})$$

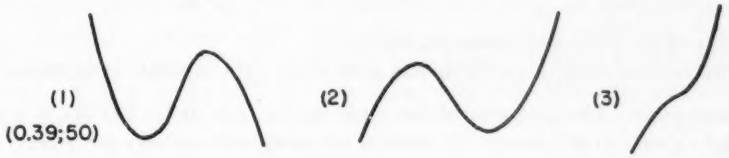
where
(0.45; 30) $A=$ (1) 4 (2) 3 (3) 1 (4) none of the proposed answers is correct,
(0.55; 33) $B=$ (1) 3 (2) 2 (3) 4 (4) none of the proposed answers is correct,
(0.34; 26) $C=$ (1) 0 (2) -3 (3) -2 (4) none of the proposed answers is correct,
(0.49; 54) $D=$ (1) 0 (2) -20 (3) -9 (4) none of the proposed answers is correct.

It should be noted that for any single problem there is considerable variation in the validity index and difficulty from response to response. This suggests the possibility of experimental investigations to determine the factors which are operating to cause these significant differences.

5. Curve sketching. Although no satisfactory method has thus far been proposed to test the ability of a student to sketch curves by the use of multiple-choice items various kinds of problems have been used to a limited extent to test certain parts of the process. Problems asking the student to determine the symmetry, extent, intercepts, *etc.*, of different equations have been used. These items are quite satisfactory to indicate whether the student understands that particular part of curve sketching. Since it is not possible to ask the student to sketch a curve when the test is an objective test using multiple-choice items it was believed that problems which require the recognition of the correct curve for a given equation and the correct equation for a given graph of a conic would help to give a more complete picture of the students' ability to sketch curves if used in conjunction with the kinds of problems previously suggested. Problems

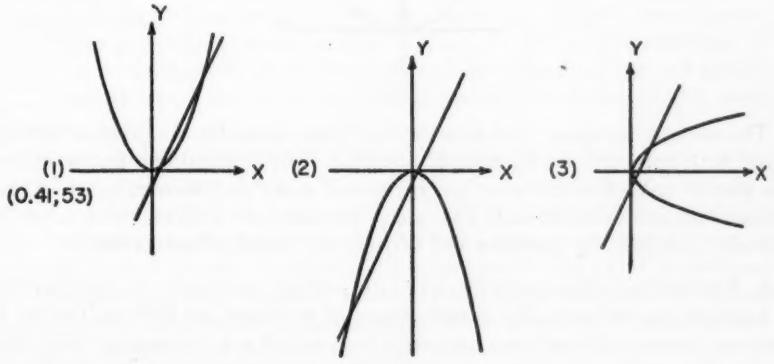
of this type were included on this test. The following examples will suggest some of the possibilities of this type of item. The statistical data included indicates the general effectiveness of such items in terms of their validity and difficulty.

Example 7. The graph of the equation $y=x^3+x^2-2x$ has the general shape



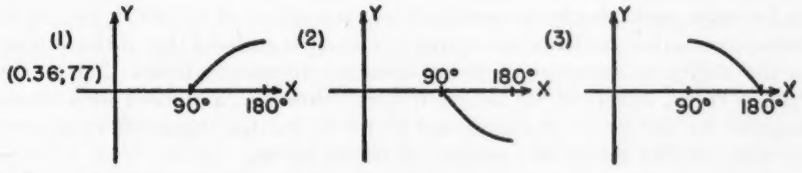
(4) none of the proposed answers is correct.

Example 8. The area bounded by the curves $y=-x^2$, and $4x-2y=1$ is sketched in



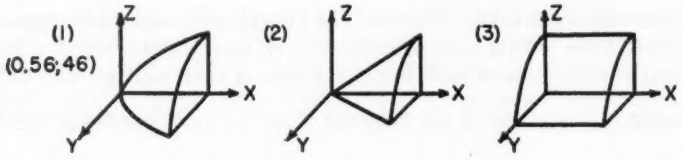
(4) none of the proposed answers is correct.

Example 9. The graph of $y=\sin x$ for values of x between 90° and 180° has the general shape



(4) none of the proposed answers is correct.

Example 10. The volume in the first octant bounded by the surfaces $x^2-y^2-z^2=0$, $x=4$, the xy-plane and the xz-plane has the general form



(4) none of the proposed answers is correct.

The reverse type of problem was also tried. One example is included.

Example 11. The equation of the conic in Fig. 2 is (0.42; 53) (1) $x^2+y^2=5$ (2) $2y^2-x=0$ (3) $x^2+4y=0$ (4) none of the proposed answers is correct.

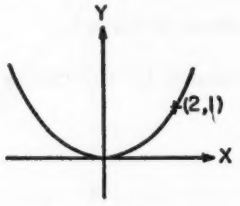


FIG. 2

The authors recognize that in several of these examples the problems are not stated with complete mathematical precision. The determining factor in each case was to make the statement as precise and exact as it seemed possible to do without confusing the student. This policy is consistent, it is believed, with that generally practiced by teachers and texts at this level of instruction.

6. Summary. In this paper we have suggested a new form of objective item for multiple-choice tests. For certain types of problems we believe it to be superior to the conventional item because it is so stated as to encourage the student to solve the problem, it does not permit the student to determine readily the answer by substitution, and at the same time it tends to minimize guessing. Since the items have such a universally high validity index for this sample their use is justified pending further investigation.

The latter part of the paper was devoted to indicating some of the possibilities for using multiple-choice questions for recognition of curves in helping to determine whether students can sketch curves. It is realized that if the student has the ability to recognize curves it does not necessarily follow that he can sketch a curve. However, we believe it does follow that a student who cannot recognize the correct curve also cannot sketch it. To that degree curve recognition does test the ability of a student to sketch curves.

The high validity, however, of these various types of items for this particular sample indicates that in so far as they do measure the ability of the student to do certain problems they discriminate well. It is hoped that others will find these suggestions helpful in constructing objective tests that are reliable and which measure more closely the desired objectives.

GEOMETRY AND EMPIRICAL SCIENCE

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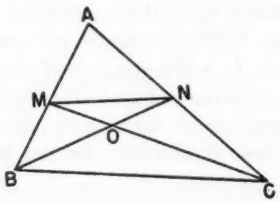
1. Introduction. The most distinctive characteristic which differentiates mathematics from the various branches of empirical science, and which accounts for its fame as the queen of the sciences, is no doubt the peculiar certainty and necessity of its results. No proposition in even the most advanced parts of empirical science can ever attain this status; a hypothesis concerning "matters of empirical fact" can at best acquire what is loosely called a high probability or a high degree of confirmation on the basis of the relevant evidence available; but however well it may have been confirmed by careful tests, the possibility can never be precluded that it will have to be discarded later in the light of new and disconfirming evidence. Thus, all the theories and hypotheses of empirical science share this provisional character of being established and accepted "until further notice," whereas a mathematical theorem, once proved, is established once and for all; it holds with that particular certainty which no subsequent empirical discoveries, however unexpected and extraordinary, can ever affect to the slightest extent. It is the purpose of this paper to examine the nature of that proverbial "mathematical certainty" with special reference to geometry, in an attempt to shed some light on the question as to the validity of geometrical theories, and their significance for our knowledge of the structure of physical space.

The nature of mathematical truth can be understood through an analysis of the method by means of which it is established. On this point I can be very brief: it is the method of mathematical demonstration, which consists in the logical deduction of the proposition to be proved from other propositions, previously established. Clearly, this procedure would involve an infinite regress unless some propositions were accepted without proof; such propositions are indeed found in every mathematical discipline which is rigorously developed; they are the *axioms* or *postulates* (we shall use these terms interchangeably) of the theory. Geometry provides the historically first example of the axiomatic presentation of a mathematical discipline. The classical set of postulates, however, on which Euclid based his system, has proved insufficient for the deduction of the well-known theorems of so-called euclidean geometry; it has therefore been revised and supplemented in modern times, and at present various adequate systems of postulates for euclidean geometry are available; the one most closely related to Euclid's system is probably that of Hilbert.

2. The inadequacy of Euclid's postulates. The inadequacy of Euclid's own set of postulates illustrates a point which is crucial for the axiomatic method in modern mathematics: Once the postulates for a theory have been laid down, every further proposition of the theory must be proved exclusively by logical deduction from the postulates; any appeal, explicit or implicit, to a feeling of self-evidence, or to the characteristics of geometrical figures, or to our experiences concerning the behavior of rigid bodies in physical space, or the like, is

strictly prohibited; such devices may have a heuristic value in guiding our efforts to find a strict proof for a theorem, but the proof itself must contain absolutely no reference to such aids. This is particularly important in geometry, where our so-called intuition of geometrical relationships, supported by reference to figures or to previous physical experiences, may induce us tacitly to make use of assumptions which are neither formulated in our postulates nor provable by means of them. Consider, for example, the theorem that in a triangle the three medians bisecting the sides intersect in one point which divides each of them in the ratio of 1:2. To prove this theorem, one shows first that in any triangle ABC (see figure) the line segment MN which connects the centers of AB and AC is parallel to BC and therefore half as long as the latter side. Then the lines BN and CM are drawn, and an examination of the triangles MON and BOC leads to the proof of the theorem. In this procedure, it is usually taken for granted that BN and CM intersect in a point O which lies between B and N as well as between C



and M. This assumption is based on geometrical intuition, and indeed, it cannot be deduced from Euclid's postulates; to make it strictly demonstrable and independent of any reference to intuition, a special group of postulates has been added to those of Euclid; they are the postulates of order. One of these—to give an example—asserts that if A, B, C are points on a straight line l, and if B lies between A and C, then B also lies between C and A.—Not even as "trivial" an assumption as this may be taken for granted; the system of postulates has to be made so complete that all the required propositions can be deduced from it by purely logical means.

Another illustration of the point under consideration is provided by the proposition that triangles which agree in two sides and the enclosed angle, are congruent. In Euclid's Elements, this proposition is presented as a theorem; the alleged proof, however, makes use of the ideas of motion and superimposition of figures and thus involves tacit assumptions which are based on our geometric intuition and on experiences with rigid bodies, but which are definitely not warranted by—*i.e.* deducible from—Euclid's postulates. In Hilbert's system, therefore, this proposition (more precisely: part of it) is explicitly included among the postulates.

3. Mathematical certainty. It is this purely deductive character of mathematical proof which forms the basis of mathematical certainty: What the rigorous proof of a theorem—say the proposition about the sum of the angles in a

triangle—establishes is not the truth of the proposition in question but rather a conditional insight to the effect that that proposition is certainly true *provided that* the postulates are true; in other words, the proof of a mathematical proposition establishes the fact that the latter is logically implied by the postulates of the theory in question. Thus, each mathematical theorem can be cast into the form

$$(P_1 \cdot P_2 \cdot P_3 \cdot \cdots \cdot P_N) \rightarrow T$$

where the expression on the left is the conjunction (joint assertion) of all the postulates, the symbol on the right represents the theorem in its customary formulation, and the arrow expresses the relation of logical implication or entailment. Precisely this character of mathematical theorems is the reason for their peculiar certainty and necessity, as I shall now attempt to show.

It is typical of any purely logical deduction that the conclusion to which it leads simply re-asserts (a proper or improper) part of what has already been stated in the premises. Thus, to illustrate this point by a very elementary example, from the premise, "This figure is a right triangle," we can deduce the conclusion, "This figure is a triangle"; but this conclusion clearly reiterates part of the information already contained in the premise. Again, from the premises, "All primes different from 2 are odd" and "n is a prime different from 2," we can infer logically that n is odd; but this consequence merely repeats part (indeed a relatively small part) of the information contained in the premises. The same situation prevails in all other cases of logical deduction; and we may, therefore, say that logical deduction—which is the one and only method of mathematical proof—is a technique of conceptual analysis: it discloses what assertions are concealed in a given set of premises, and it makes us realize to what we committed ourselves in accepting those premises; but none of the results obtained by this technique ever goes by one iota beyond the information already contained in the initial assumptions.

Since all mathematical proofs rest exclusively on logical deductions from certain postulates, it follows that a mathematical theorem, such as the Pythagorean theorem in geometry, asserts nothing that is *objectively* or *theoretically new* as compared with the postulates from which it is derived, although its content may well be *psychologically new* in the sense that we were not aware of its being implicitly contained in the postulates.

The nature of the peculiar certainty of mathematics is now clear: A mathematical theorem is certain *relatively* to the set of postulates from which it is derived; *i.e.* it is necessarily true *if* those postulates are true; and this is so because the theorem, if rigorously proved, simply re-asserts part of what has been stipulated in the postulates. A truth of this conditional type obviously implies no assertions about matters of empirical fact and can, therefore, never get into conflict with any empirical findings, even of the most unexpected kind; consequently, unlike the hypotheses and theories of empirical science, it can never suffer the fate of being disconfirmed by new evidence: A mathematical truth is

irrefutably certain just because it is devoid of factual, or empirical content. Any theorem of geometry, therefore, when cast into the conditional form described earlier, is analytic in the technical sense of logic, and thus true *a priori*; *i.e.* its truth can be established by means of the formal machinery of logic alone, without any reference to empirical data.

4. Postulates and truth. Now it might be felt that our analysis of geometrical truth so far tells only half of the relevant story. For while a geometrical proof no doubt enables us to assert a proposition conditionally—namely on condition that the postulates are accepted—, is it not correct to add that geometry also unconditionally asserts the truth of its postulates and thus, by virtue of the deductive relationship between postulates and theorems, enables us unconditionally to assert the truth of its theorems? Is it not an unconditional assertion of geometry that two points determine one and only one straight line that connects them, or that in any triangle, the sum of the angles equals two right angles? That this is definitely not the case, is evidenced by two important aspects of the axiomatic treatment of geometry which will now be briefly considered.

The first of these features is the well-known fact that in the more recent development of mathematics, several systems of geometry have been constructed which are incompatible with euclidean geometry, and in which, for example, the two propositions just mentioned do not necessarily hold. Let us briefly recollect some of the basic facts concerning these *non-euclidean geometries*. The postulates on which euclidean geometry rests include the famous postulate of the parallels, which, in the case of plane geometry, asserts in effect that through every point P not on a given line l there exists exactly one parallel to l, *i.e.*, one straight line which does not meet l. As this postulate is considerably less simple than the others, and as it was also felt to be intuitively less plausible than the latter, many efforts were made in the history of geometry to prove that this proposition need not be accepted as an axiom, but that it can be deduced as a theorem from the remaining body of postulates. All attempts in this direction failed, however; and finally it was conclusively demonstrated that a proof of the parallel principle on the basis of the other postulates of euclidean geometry (even in its modern, completed form) is impossible. This was shown by proving that a perfectly self-consistent geometrical theory is obtained if the postulate of the parallels is replaced by the assumption that through any point P not on a given straight line l there exist at least two parallels to l. This postulate obviously contradicts the euclidean postulate of the parallels, and if the latter were actually a consequence of the other postulates of euclidean geometry, then the new set of postulates would clearly involve a contradiction, which can be shown not to be the case. This first non-euclidean type of geometry, which is called hyperbolic geometry, was discovered in the early 20's of the last century almost simultaneously, but independently by the Russian N. I. Lobatschefskij, and by the Hungarian J. Bolyai. Later, Riemann developed an alternative geometry, known as elliptical geometry, in which the axiom of the parallels is replaced by the postulate that no line has any parallels. (The acceptance of this postulate,

however, in contradistinction to that of hyperbolic geometry, requires the modification of some further axioms of euclidean geometry, if a consistent new theory is to result.) As is to be expected, many of the theorems of these non-euclidean geometries are at variance with those of euclidean theory; thus, *e.g.*, in the hyperbolic geometry of two dimensions, there exist, for each straight line l, through any point P not on l, infinitely many straight lines which do not meet l; also, the sum of the angles in any triangle is less than two right angles. In elliptic geometry, this angle sum is always greater than two right angles; no two straight lines are parallel; and while two different points usually determine exactly one straight line connecting them (as they always do in euclidean geometry), there are certain pairs of points which are connected by infinitely many different straight lines. An illustration of this latter type of geometry is provided by the geometrical structure of that curved two-dimensional space which is represented by the surface of a sphere, when the concept of straight line is interpreted by that of great circle on the sphere. In this space, there are no parallel lines since any two great circles intersect; the endpoints of any diameter of the sphere are points connected by infinitely many different "straight lines," and the sum of the angles in a triangle is always in excess of two right angles. Also, in this space, the ratio between the circumference and the diameter of a circle (not necessarily a great circle) is always less than 2π.

Elliptic and hyperbolic geometry are not the only types of non-euclidean geometry; various other types have been developed; we shall later have occasion to refer to a much more general form of non-euclidean geometry which was likewise devised by Riemann.

The fact that these different types of geometry have been developed in modern mathematics shows clearly that mathematics cannot be said to assert the truth of any particular set of geometrical postulates; all that pure mathematics is interested in, and all that it can establish, is the deductive consequences of given sets of postulates and thus the necessary truth of the ensuing theorems relatively to the postulates under consideration.

A second observation which likewise shows that mathematics does not assert the truth of any particular set of postulates refers to *the status of the concepts in geometry*. There exists, in every axiomatized theory, a close parallelism between the treatment of the propositions and that of the concepts of the system. As we have seen, the propositions fall into two classes: the postulates, for which no proof is given, and the theorems, each of which has to be derived from the postulates. Analogously, the concepts fall into two classes: the primitive or basic concepts, for which no definition is given, and the others, each of which has to be precisely defined in terms of the primitives. (The admission of some undefined concepts is clearly necessary if an infinite regress in definition is to be avoided.) The analogy goes farther: Just as there exists an infinity of theoretically suitable axiom systems for one and the same theory—say, euclidean geometry—, so there also exists an infinity of theoretically possible choices for the primitive terms of that theory; very often—but not always—different axiomatizations of the same theory involve not only different postulates, but also differ-

ent sets of primitives. Hilbert's axiomatization of plane geometry contains six primitives: point, straight line, incidence (of a point on a line), betweenness (as a relation of three points on a straight line), congruence for line segments, and congruence for angles. (Solid geometry, in Hilbert's axiomatization, requires two further primitives, that of plane and that of incidence of a point on a plane.) All other concepts of geometry, such as those of angle, triangle, circle, *etc.*, are defined in terms of these basic concepts.

But if the primitives are not defined within geometrical theory, what meaning are we to assign to them? The answer is that it is entirely unnecessary to connect any particular meaning with them. True, the words "point," "straight line," *etc.*, carry definite connotations with them which relate to the familiar geometrical figures, but the validity of the propositions is completely independent of these connotations. Indeed, suppose that in axiomatized euclidean geometry, we replace the over-suggestive terms "point," "straight line," "incidence," "betweenness," *etc.*, by the neutral terms "object of kind 1," " object of kind 2," "relation No. 1," "relation No. 2," *etc.*, and suppose that we present this modified wording of geometry to a competent mathematician or logician who, however, knows nothing of the customary connotations of the primitive terms. For this logician, all proofs would clearly remain valid, for as we saw before, a rigorous proof in geometry rests on deduction from the axioms alone without any reference to the customary interpretation of the various geometrical concepts used. We see therefore that indeed no specific meaning has to be attached to the primitive terms of an axiomatized theory; and in a precise logical presentation of axiomatized geometry the primitive concepts are accordingly treated as so-called logical variables.

As a consequence, geometry cannot be said to assert the truth of its postulates, since the latter are formulated in terms of concepts without any specific meaning; indeed, for this very reason, the postulates themselves do not make any specific assertion which could possibly be called true or false! In the terminology of modern logic, the postulates are not sentences, but sentential functions with the primitive concepts as variable arguments.—This point also shows that the postulates of geometry cannot be considered as "self-evident truths," because where no assertion is made, no self-evidence can be claimed.

5. Pure and physical geometry. Geometry thus construed is a purely formal discipline; we shall refer to it also as *pure geometry*. A pure geometry, then,—no matter whether it is of the euclidean or of a non-euclidean variety—deals with no specific subject-matter; in particular, it asserts nothing about physical space. All its theorems are analytic and thus true with certainty precisely because they are devoid of factual content. Thus, to characterize the import of pure geometry, we might use the standard form of a movie-disclaimer: No portrayal of the characteristics of geometrical figures or of the spatial properties or relationships of actual physical bodies is intended, and any similarities between the primitive concepts and their customary geometrical connotations are purely coincidental.

But just as in the case of some motion pictures, so in the case at least of euclidean geometry, the disclaimer does not sound quite convincing: Historically speaking, at least, euclidean geometry has its origin in the generalization and systematization of certain empirical discoveries which were made in connection with the measurement of areas and volumes, the practice of surveying, and the development of astronomy. Thus understood, geometry has factual import; it is an empirical science which might be called, in very general terms, the theory of the structure of physical space, or briefly, *physical geometry*. What is the relation between pure and physical geometry?

When the physicist uses the concepts of point, straight line, incidence, *etc.*, in statements about physical objects, he obviously connects with each of them a more or less definite physical meaning. Thus, the term "point" serves to designate physical points, *i.e.*, objects of the kind illustrated by pin-points, cross hairs, *etc.* Similarly, the term "straight line" refers to straight lines in the sense of physics, such as illustrated by taut strings or by the path of light rays in a homogeneous medium. Analogously, each of the other geometrical concepts has a concrete physical meaning in the statements of physical geometry. In view of this situation, we can say that physical geometry is obtained by what is called, in contemporary logic, a semantical interpretation of pure geometry, Generally speaking, a semantical interpretation of a pure mathematical theory, whose primitives are not assigned any specific meaning, consists in giving each primitive (and thus, indirectly, each defined term) a specific meaning or designatum. In the case of physical geometry, this meaning is physical in the sense just illustrated; it is possible, however, to assign a purely arithmetical meaning to each concept of geometry; the possibility of such an arithmetical interpretation of geometry is of great importance in the study of the consistency and other logical characteristics of geometry, but it falls outside the scope of the present discussion.

By virtue of the physical interpretation of the originally uninterpreted primitives of a geometrical theory, physical meaning is indirectly assigned also to every defined concept of the theory; and if every geometrical term is now taken in its physical interpretation, then every postulate and every theorem of the theory under consideration turns into a statement of physics, with respect to which the question as to truth or falsity may meaningfully be raised—a circumstance which clearly contradistinguishes the propositions of physical geometry from those of the corresponding uninterpreted pure theory.—Consider, for example, the following postulate of pure euclidean geometry: For any two objects x, y of kind 1, there exists exactly one object l of kind 2 such that both x and y stand in relation No. 1 to l. As long as the three primitives occurring in this postulate are uninterpreted, it is obviously meaningless to ask whether the postulate is true. But by virtue of the above physical interpretation, the postulate turns into the following statement: For any two physical points x, y there exists exactly one physical straight line l such that both x and y lie on l. But this is a physical hypothesis, and we may now meaningfully ask whether it is true or

false. Similarly, the theorem about the sum of the angles in a triangle turns into the assertion that the sum of the angles (in the physical sense) of a figure bounded by the paths of three light rays equals two right angles.

Thus, the physical interpretation transforms a given pure geometrical theory—euclidean or non-euclidean—into a system of physical hypotheses which, if true, might be said to constitute a theory of the structure of physical space. But the question whether a given geometrical theory in physical interpretation is factually correct represents a problem not of pure mathematics but of empirical science; it has to be settled on the basis of suitable experiments or systematic observations. The only assertion the mathematician can make in this context is this: If all the postulates of a given geometry, in their physical interpretation, are true, then all the theorems of that geometry, in their physical interpretation, are necessarily true, too, since they are logically deducible from the postulates. It might seem, therefore, that in order to decide whether physical space is euclidean or non-euclidean in structure, all that we have to do is to test the respective postulates in their physical interpretation. However, this is not directly feasible; here, as in the case of any other physical theory, the basic hypotheses are largely incapable of a direct experimental test; in geometry, this is particularly obvious for such postulates as the parallel axiom or Cantor's axiom of continuity in Hilbert's system of euclidean geometry, which makes an assertion about certain infinite sets of points on a straight line. Thus, the empirical test of a physical geometry no less than that of any other scientific theory has to proceed indirectly; namely, by deducing from the basic hypotheses of the theory certain consequences, or predictions, which are amenable to an experimental test. If a test bears out a prediction, then it constitutes confirming evidence (though, of course, no conclusive proof) for the theory; otherwise, it disconfirms the theory. If an adequate amount of confirming evidence for a theory has been established, and if no disconfirming evidence has been found, then the theory may be accepted by the scientist "until further notice."

It is in the context of this indirect procedure that pure mathematics and logic acquire their inestimable importance for empirical science: While formal logic and pure mathematics do not in themselves establish any assertions about matters of empirical fact, they provide an efficient and entirely indispensable machinery for deducing, from abstract theoretical assumptions, such as the laws of Newtonian mechanics or the postulates of euclidean geometry in physical interpretation, consequences concrete and specific enough to be accessible to direct experimental test. Thus, *e.g.*, pure euclidean geometry shows that from its postulates there may be deduced the theorem about the sum of the angles in a triangle, and that this deduction is possible no matter how the basic concepts of geometry are interpreted; hence also in the case of the physical interpretation of euclidean geometry. This theorem, in its physical interpretation, is accessible to experimental test; and since the postulates of elliptic and of hyperbolic geometry imply values different from two right angles for the angle sum of a triangle, this particular proposition seems to afford a good opportunity for a crucial experi-

ment. And no less a mathematician than Gauss did indeed perform this test; by means of optical methods—and thus using the interpretation of physical straight lines as paths of light rays—he ascertained the angle sum of a large triangle determined by three mountain tops. Within the limits of experimental error, he found it equal to two right angles.

6. On Poincaré's conventionalism concerning geometry. But suppose that Gauss had found a noticeable deviation from this value; would that have meant a refutation of euclidean geometry in its physical interpretation, or, in other words, of the hypothesis that physical space is euclidean in structure? Not necessarily; for the deviation might have been accounted for by a hypothesis to the effects that the paths of the light rays involved in the sighting process were bent by some disturbing force and thus were not actually straight lines. The same kind of reference to deforming forces could also be used if, say, the euclidean theorems of congruence for plane figures were tested in their physical interpretation by means of experiments involving rigid bodies, and if any violations of the theorems were found. This point is by no means trivial; Henri Poincaré, the great French mathematician and theoretical physicist, based on considerations of this type his famous *conventionalism concerning geometry*. It was his opinion that no empirical test, whatever its outcome, can conclusively invalidate the euclidean conception of physical space; in other words, the validity of euclidean geometry in physical science can always be preserved—if necessary, by suitable changes in the theories of physics, such as the introduction of new hypotheses concerning deforming or deflecting forces. Thus, the question as to whether physical space has a euclidean or a non-euclidean structure would become a matter of convention, and the decision to preserve euclidean geometry at all costs would recommend itself, according to Poincaré, by the greater simplicity of euclidean as compared with non-euclidean geometrical theory.

It appears, however, that Poincaré's account is an oversimplification. It rightly calls attention to the fact that the test of a physical geometry G always presupposes a certain body P of non-geometrical physical hypotheses (including the physical theory of the instruments of measurement and observation used in the test), and that the so-called test of G actually bears on the combined theoretical system $G \cdot P$ rather than on G alone. Now, if predictions derived from $G \cdot P$ are contradicted by experimental findings, then a change in the theoretical structure becomes necessary. In classical physics, G always was euclidean geometry in its physical interpretation, GE; and when experimental evidence required a modification of the theory, it was P rather than GE which was changed. But Poincaré's assertion that this procedure would always be distinguished by its greater simplicity is not entirely correct; for what has to be taken into consideration is the simplicity of the total system $G \cdot P$, and not just that of its geometrical part. And here it is clearly conceivable that a simpler total theory in accordance with all the relevant empirical evidence is obtainable by going over to a non-euclidean form of geometry rather than by preserving the euclidean structure of physical space and making adjustments only in part P.

And indeed, just this situation has arisen in physics in connection with the development of the general theory of relativity: If the primitive terms of geometry are given physical interpretations along the lines indicated before, then certain findings in astronomy represent good evidence in favor of a total physical theory with a non-euclidean geometry as part *G*. According to this theory, the physical universe at large is a three-dimensional curved space of a very complex geometrical structure; it is finite in volume and yet unbounded in all directions. However, in comparatively small areas, such as those involved in Gauss' experiment, euclidean geometry can serve as a good approximative account of the geometrical structure of space. The kind of structure ascribed to physical space in this theory may be illustrated by an analogue in two dimensions; namely, the surface of a sphere. The geometrical structure of the latter, as was pointed out before, can be described by means of elliptic geometry, if the primitive term "straight line" is interpreted as meaning "great circle," and if the other primitives are given analogous interpretations. In this sense, the surface of a sphere is a two-dimensional curved space of non-euclidean structure, whereas the plane is a two-dimensional space of euclidean structure. While the plane is unbounded in all directions, and infinite in size, the spherical surface is finite in size and yet unbounded in all directions: a two-dimensional physicist, travelling along "straight lines" of that space would never encounter any boundaries of his space; instead, he would finally return to his point of departure, provided that his life span and his technical facilities were sufficient for such a trip in consideration of the size of his "universe." It is interesting to note that the physicists of that world, even if they lacked any intuition of a three-dimensional space, could empirically ascertain the fact that their two-dimensional space was curved. This might be done by means of the method of traveling along straight lines; another, simpler test would consist in determining the angle sum in a triangle; again another in determining, by means of measuring tapes, the ratio of the circumference of a circle (not necessarily a great circle) to its diameter; this ratio would turn out to be less than π.

The geometrical structure which relativity physics ascribes to physical space is a three-dimensional analogue to that of the surface of a sphere, or, to be more exact, to that of the closed and finite surface of a potato, whose curvature varies from point to point. In our physical universe, the curvature of space at a given point is determined by the distribution of masses in its neighborhood; near large masses such as the sun, space is strongly curved, while in regions of low mass-density, the structure of the universe is approximately euclidean. The hypothesis stating the connection between the mass distribution and the curvature of space at a point has been approximately confirmed by astronomical observations concerning the paths of light rays in the gravitational field of the sun.

The geometrical theory which is used to describe the structure of the physical universe is of a type that may be characterized as a generalization of elliptic geometry. It was originally constructed by Riemann as a purely mathematical theory, without any concrete possibility of practical application at hand. When

Einstein, in developing his general theory of relativity, looked for an appropriate mathematical theory to deal with the structure of physical space, he found in Riemann's abstract system the conceptual tool he needed. This fact throws an interesting sidelight on the importance for scientific progress of that type of investigation which the "practical-minded" man in the street tends to dismiss as useless, abstract mathematical speculation.

Of course, a geometrical theory in physical interpretation can never be validated with mathematical certainty, no matter how extensive the experimental tests to which it is subjected; like any other theory of empirical science, it can acquire only a more or less high degree of confirmation. Indeed, the considerations presented in this article show that the demand for mathematical certainty in empirical matters is misguided and unreasonable; for, as we saw, mathematical certainty of knowledge can be attained only at the price of analyticity and thus of complete lack of factual content. Let me summarize this insight in Einstein's words:

"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."

FUNCTIONS OF SEVERAL COMPLEX VARIABLES*

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1. Definition of an analytic function. Consider a domain D in the $2n$-dimensional euclidean space of n complex variables $z_1, \cdots, z_n$. A function $f(z_1, \cdots, z_n)$ is said to be analytic in D if in some neighborhood of every point $(z_1, \cdots, z_n)$ of D it can be represented as the sum of an (absolutely) convergent multiple power-series

$$\sum_{i_1, \cdots, i_n=0} a_{i_1, \cdots, i_n}(z_1 - z_1^0)^{i_1} \cdots (z_n - z_n^0)^{i_n}.$$

An alternative definition is that f is analytic in D if it has derivatives of all orders, mixed and iterated, at every point of D. These two definitions are very easily seen to be equivalent.

An important result due to Osgood [1] in 1899 states that *if $f(z_1, \cdots, z_n)$ is bounded in D and if the n partial derivatives $\partial f/\partial z_j$, $j=1, \cdots, n$ all exist at every point of D, then f is analytic in D.*

In 1899 and again in 1900 Osgood [1, 2] raised the question as to whether the boundedness restriction could be removed, and in 1906 Hartogs [1] showed that it actually could be removed. This means that if a function is analytic in each variable separately, it is analytic as a function of all n variables. This is a key result in the theory.

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2. Some general remarks. Roughly the results of analytic functions of several complex variables can be divided into two main groups, those results which are generalizations of the results of one complex variable, and those results which are peculiar to several variables, either in that they have no analogues in one variable or in that their analogues in one variable are false. It is necessary and important that both types of results be investigated. Often it is extremely difficult to carry a result over from one to several variables, and completely new ideas are needed. Sometimes these new ideas even clarify ideas in one variable. In this particular talk I want to limit myself to a few examples of the second type, that is to a few results which are more or less peculiar to several variables. One such result is the key theorem of Hartogs mentioned above—this theorem has no meaning in one variable and hence is truly a result on several variables. As a matter of fact it is a result peculiar to several *complex* variables since it is false when the variables are real, as one sees by considering the example

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & 0 < x^2 + y^2, \\ 0, & x = y = 0, \end{cases}$$

in a neighborhood of the origin (*cf.* Osgood [3, p. 142]). In proving the theorem Hartogs was led to the introduction of subharmonic functions, a topic which has proved very fruitful in other parts of analysis.

I will not dwell further here on this theorem but will pass to a second example which also bears the name of Hartogs. It is perhaps interesting to remark that Hartogs' key theorem given above involves mainly the work of two men, Osgood and Hartogs, and was proved in 1906; while the theorem to be considered next has been contributed to by many mathematicians over the period of the past four decades.

3. Hartogs' theorem on analytic continuation from the boundary of a domain. An analytic function of one complex variable in a domain D has isolated zeros in D and its reciprocal has isolated singularities (poles) in D. This situation no longer prevails in several variables—an analytic function of several complex variables cannot have isolated zeros or singularities (except for trivial removable singularities). For example, the function $1/z_1$ has as singularities all points of the $(2n-2)$-dimensional set $z_1 = 0$; $z_2, \cdots, z_n$ arbitrary. This property, as well as other considerations, led mathematicians early to state roughly that "the singularities of an analytic function of several complex variables must penetrate the boundary." This idea is rigorously embodied in the following theorem:

HARTOGS' THEOREM. *Let D be a bounded domain in $(z_1, \cdots, z_n)$-space $(n>1)$ with a connected boundary C, and let $f(z_1, \cdots, z_n)$ be a single-valued function which is analytic in some domain containing C. Then f has a unique analytic continuation into all of D.*

The hypothesis $n>1$ of the theorem is necessary—every domain in the plane is the (exact) existence domain of an analytic function of one complex variable.

Hartogs [2] proved the theorem in 1906 for the case in which D is a cylindrical domain, that is, the topological product $[z_1 \in D_1, \cdots, z_n \in D_n]$, where D_i is a domain in the z_i-plane. He did this by repeated use of Cauchy's integral formula for one variable. Since 1906 the theorem has attracted various mathematicians.

There are three main methods of attack on the theorem. The first method uses Cauchy's integral formula for one variable to yield an important lemma concerning continuation from the boundary of a hypersphere (Lemma 1 below), and then bases the proof upon this lemma. The second method is closely related; it again uses Cauchy's integral formula for one variable, treating the other variables as parameters. The third method differs importantly from the other two in that it uses a recent form of Cauchy's integral formula for n variables taken over the $(2n-1)$-dimensional boundary of the domain. We shall amplify these remarks a little.

3a. The first method. The following important lemma is due essentially to Levi [1] and Osgood [3] although it is based upon Hartogs' work.

LEMMA 1. *Let P be a boundary point of a hypersphere in E_{2n} $(n>1)$ and let $f(z_1, \cdots, z_n)$ be a function which is defined and analytic in the portion of a neighborhood of P which lies outside the (closed) hypersphere. Then f has a unique analytic continuation into a (2n-dimensional) neighborhood containing P.*

The lemma is proved very readily by use of Cauchy's integral formula for one variable; the fact that n is greater than unity is of importance in allowing one to shift the contour. The idea of the proof will become evident as we consider an example in Section 3b.

Osgood [3, 4] used this lemma in attacking Hartogs' general theorem. Because of certain geometric difficulties which apparently were not fully met in Osgood's proof, Brown [1] in 1936 gave a detailed proof of the theorem, again using the above lemma. Gaps in at least one other proof by this method have been observed; suffice it to say that a proof of Hartogs' theorem by this method calls for very careful and painstaking analysis.

3b. The second method. By use of Cauchy's integral formula for one complex variable, Severi [1] in 1932 proved the analogue of Hartogs' theorem for a wide class of domains for analytic functions $f(z, u)$ of one complex and one real variable. In the paper mentioned earlier, Brown [1] extended Severi's result for functions $f(z_1, \cdots, z_n, u)$, $(n>0)$, with $(z_1, \cdots, z_n)$ complex, u real, and for general bounded domains D in E_{2n+1} with connected boundaries. A function $f(z_1, \cdots, z_n, u_1, \cdots, u_m)$ of n complex variables and m real variables is said to be an analytic function of these $n+m$ variables in a domain D of E_{2n+m} if in some neighborhood of every point of D it can be developed in an (absolutely) convergent multiple power-series.

We will illustrate Severi's method by the following example. For the sake of simplicity, we will work with one real variable.

Example. Let $f(z, u)$ be analytic in the spherical shell

$$(R-\epsilon)^2 < |z|^2 + u^2 < (R+\epsilon)^2, \qquad (0 < \epsilon < R). \tag{1}$$

Then f has an analytic continuation into the sphere

$$|z|^2 + u^2 < (R+\epsilon)^2. \tag{2}$$

Proof. For each fixed value of u in

$$-R < u < R, \tag{3}$$

define

$$\phi(z, u) = \frac{1}{2\pi i}\int_{C(u)} \frac{f(t, u)}{t - z}\, dt, \tag{4}$$

where $C(u)$ is the circle

$$|t| = [R^2 - u^2]^{1/2}. \tag{5}$$

Since $f(t, u)$ is a continuous (even analytic) function of t for t on the contour $C(u)$ it follows that $\phi(z, u)$ is an analytic function of z for z in the interior of the circle $C(u)$,

$$|z| < [R^2 - u^2]^{1/2}. \tag{6}$$

We next observe that since $f(t, u)$ is analytic in the shell (1) we may shift the contour in (4) to any simple closed contour contained within the annulus

$$[(R-\epsilon)^2 - u^2]^{1/2} < |t| < [(R+\epsilon)^2 - u^2]^{1/2} \tag{7}$$

(u fixed in (3)), and not change the value of the resulting function $\phi(z, u)$. We will use this fact a little later.

Now let u_0 be a fixed value of u in (3),

$$-R < u_0 < R, \tag{8}$$

and let us form the integral

$$\frac{1}{2\pi i}\int_{C(u_0)} \frac{f(t, u)}{t - z}\, dt, \tag{9}$$

where the contour $C(u_0)$ is fixed but where we now let u vary over an interval

$$u_0 - \epsilon < u < u_0 + \epsilon. \tag{10}$$

Since $f(z, u)$ is given to be analytic in (z, u) in the shell (1), and since the point set

$$[t \in C(u_0),\ u_0 - \epsilon < u < u_0 + \epsilon] \tag{11}$$

lies in the shell (1), it follows easily that the integral (9) defines a function which is analytic in (z, u) in the domain

$$[|z|^2 < R^2 - u_0^2,\ u_0 - \epsilon < u < u_0 + \epsilon]. \tag{12}$$

We now make two simple observations. First, the function (9) and the function $\phi(z, u)$ are identical for $u = u_0$, as one sees by comparing their definitions. Secondly, by the remark made in the preceding paragraph, the function (9) and the function $\phi(z, u)$ actually agree for all values of u in (10) since for such values of u the contour $C(u_0)$ serves as an admissible contour. Hence $\phi(z, u)$ is analytic in (z, u) throughout the entire sphere $|z|^2+u^2<R^2$.

It remains to identify ϕ with f. This is very easily done. Consider for example a value of u in, say,

$$R - \epsilon < u < R - \frac{\epsilon}{2}.$$

For such a value of u the function $f(t, u)$ is analytic, not just in a neighborhood of the contour $C(u)$ but even throughout its interior. Hence for such values of u we have $\phi(z, u) \equiv f(z, u)$, and hence $\phi(z, u)$ furnishes the desired analytic continuation of $f(z, u)$ into $|z|^2+u^2<R^2$. This yields the desired result.

This example illustrates Severi's approach; by such methods he proved Hartogs' theorem for a wide class of domains for functions of one complex and one real variable. In addition to the work of Brown mentioned earlier, further work in this direction is found in a manuscript by Bochner and the speaker [1].

3c. The third method. As mentioned earlier this method is rather recent and uses an integral formula for analytic functions taken over the $(2n-1)$-dimensional boundary of a domain. The corresponding theorem for analytic functions of quaternions was proved by Fueter [1] in 1939 by this method; this yielded Hartogs' theorem in its general form for $n=2$. Bochner [1, 2], Fueter [2], Martinelli [1] and May [1] have proved the result for general domains and for general values of $n \geqq 2$ by this method. The papers of Fueter and Martinelli are closely related and independent of the work of Bochner and his student, May. The Fueter approach is from the standpoint of quaternions and other Cayley number systems, while Bochner and May approach the problem directly from the standpoint of analytic functions of several complex variables. We shall now give an outline of Bochner's approach.

Given a bounded domain D in E_{2n} $(n>1)$ with connected boundary C, and given a function $f(z_1, \cdots, z_n)$ defined and analytic in a neighborhood N of C, define a function

$$\phi(z_1, \cdots, z_n) = (-1)^n \frac{(n-1)!}{(2\pi i)^n} \int_C \frac{f(t_1, \cdots, t_n)}{[\sum_{j=1}^n |t_j - z_j|^2]^n} \sum_{j=1}^n \overline{t_j - z_j}[dt]^{(j)},$$

where $[dt]^{(j)}$ is a $(2n-1)$-dimensional element on C which can be defined as follows in terms of the Jacobian:

$$[dt]^{(j)} = \frac{\partial(t_1, \bar{t}_1, \cdots, t_{j-1}, \bar{t}_{j-1}, t_j, t_{j+1}, \bar{t}_{j+1}, \cdots, t_n, \bar{t}_n)}{\partial(\sigma_1, \sigma_2, \cdots, \sigma_{2n-1})} d\sigma_1 \cdots d\sigma_{2n-1},$$

where $\sigma_1, \cdots, \sigma_{2n-1}$ form a system of (real) coordinates on C. (In case the boundary C of the domain D is not sufficiently smooth, it may be necessary to shift the path of integration slightly. Bochner and May have shown that this can always be done in such a way as to remain in any preassigned neighborhood N of C, *cf.* May [1].) With $\phi(z_1, \cdots, z_n)$ so defined one first shows that ϕ is analytic throughout the interior of D and one next identifies ϕ with f by considering points near the boundary C. The facts that n is greater than 1 and that D is bounded both enter very much into the proof. We shall omit the details.

This method of attack avoids the gaps which were mentioned earlier as having arisen in certain of the proofs by the first method.

3d. In concluding section 3 we merely mention that Levi [1] has carried Hartogs' theorem over to the case of meromorphic functions. The paper by Brown also includes the case of meromorphic functions.

4. The problems of Cousin and Poincaré. Among the earliest and most difficult problems are those of trying to carry over to several complex variables the results of Mittag-Leffler and Weierstrass on the existence of a meromorphic (analytic) function having given poles (zeros) and the related question on the quotient representation of a meromorphic function. Especial difficulties arise in several variables due to the fact that the zeros and poles of an analytic function of several variables are no longer isolated but are $(2n-2)$-dimensional manifolds.

Cousin's first problem is an attempt to carry over to several variables Mittag-Leffler's theorem on the existence of a meromorphic function having prescribed poles. His second problem is to carry over Weierstrass's theorem on the existence of an analytic function with given zeros. Poincaré's problem is to show that a meromorphic function in a domain D can be represented as a quotient of two functions throughout D, not just locally.

Cousin's First Problem. To every point P of a domain D let there be a neighborhood $N(P)$ and a function f_P meromorphic in $N(P)$. In the intersection $N(P) \cap N(Q)$ of the neighborhoods of two points P and Q of D let $f_P - f_Q$ be regular. Then does there always exist a single-valued meromorphic function F in D so that $F - f_P$ is regular in $N(P)$?

Cousin's Second Problem. To every point P of a domain D let there be a neighborhood $N(P)$ and a function f_P regular in $N(P)$. In the intersection $N(P) \cap N(Q)$ of the neighborhoods of two points P and Q of D let f_P/f_Q be regular and different from zero. Then does there always exist a single-valued analytic function F in D so that F/f_P is regular and different from zero in $N(P)$?

Poincaré's Problem. If a function f is single-valued and meromorphic in a domain D, do there always exist two functions g and h, analytic in D and prime to each other, so that the relation

$$h \cdot f \equiv g$$

holds throughout D?

(Two analytic functions g and h in a domain D are said to be prime to each other if to every common zero P_0 in D, there is no triple of functions d, g_1, h_1

analytic in a neighborhood of P_0, all vanishing at P_0 and such that $g \equiv d \cdot g_1$, $h = d \cdot h_1$ in $N(P_0)$.)

If for a given domain D, Cousin's first problem always can be answered in the affirmative then we say that Cousin's first assertion is valid for D. We do similarly for Cousin's second assertion and Poincaré's assertion.

Now all three assertions are valid for every *schlicht* domain in one complex variable, but the situation is entirely different for several variables. We shall see in what ways it differs as we proceed.

In 1883 Poincaré [1] for $n=2$ proved that his assertion is valid for the entire space. Later, in 1895, Cousin [1] proved that all three assertions are valid for all simply connected cylindrical domains D: $[z_1 \in D_1, \cdots, z_n \in D_n]$. Cousin claimed to prove that all three assertions are valid for all cylindrical domains, whether simply connected or not, but Gronwall [1] in 1917 found a gap in Cousin's work and showed by a counterexample that Cousin's second assertion is in general false if D is not simply connected; the same prevails for Poincaré's assertion. As far as cylindrical domains go, Cousin's first assertion is valid for all cylindrical domains while Cousin's second assertion and Poincare's assertion hold for all cylindrical domains at most one of whose components is multiply connected.

In 1934 Cartan [1] proved that for $n=2$, Cousin's first assertion can hold at most in domains which are (exact) existence domains for some analytic function. (The proof of this is given in Behnke and Stein [1].) As far as I have been able to determine it is still an open question as to whether this result of Cartan is true for $n>2$.

In 1936, 1937, Oka [1, 2] proved a result related to that of Cartan; he showed for $n \geqq 2$ that Cousin's first assertion is valid in all finite *schlicht* domains D which are existence domains for some analytic function.

Two other important results are the following:

Result 1. The validity of Cousin's second assertion for a domain implies the validity of Poincaré's assertion for that domain.

Result 2. If Cousin's first assertion and Poincaré's assertion are both valid for a domain D then Cousin's second assertion is also valid for D (Behnke-Stein [1], 1937).

Result 1 may be seen as follows (compare Behnke-Thullen [1, p. 67]). If F is meromorphic in D and P is a point of D, then by definition there exists a pair of functions of f_p and h_p, regular at P and prime to each other at P, so that

$$F = \frac{h_p}{f_p}$$

in a neighborhood of P; if F is regular at P, then $f_p \equiv 1$. These functions f_p obviously satisfy the conditions of Cousin's second problem for suitably selected neighborhoods $N(P)$. If therefore D is a domain for which Cousin's second assumption is valid then there exists a function G, regular in D, which is such that

G/f_p is regular and non-vanishing at P. Now consider the function $F \cdot G$; in the neighborhood of any point P of D it is representable in the form

$$F \cdot G = h_p \cdot \frac{G}{f_p}$$

and each of the two factors h_p and (G/f_p) is analytic in a neighborhood of P. Hence $F \cdot G$ is analytic throughout D; that is, there is a function H, analytic in D, such that

$$F \cdot G = H$$

holds throughout D; also G and H are prime to each other in any common zero.

This yields Result 1. We will not give the proof of Result 2.

Behnke and Stein [1] have made a very interesting analysis of the relationship of the three problems under consideration. We shall next give a brief résumé of their analysis and in the next section shall discuss briefly two other important contributions related to the work of this section.

If the three problems were entirely independent of each other there would be $2^3 = 8$ possibilities for a given domain. Actually they are not entirely independent since we have seen (Results 1 and 2 of this section) that Cousin's second assertion for a given domain implies Poincaré's, while Cousin's first assertion with Poincaré's assertion implies Cousin's second assertion. Therefore there are at most five possibilities for a given domain; Behnke and Stein have shown these five possibilities all actually do occur. They have given the following table:

	Cousin I	*Cousin II*	*Poincaré*	*Possible?*
1	+	+	+	+
2	+	+	−	−
3	+	−	+	−
4	+	−	−	+
5	−	+	+	+
6	−	+	−	−
7	−	−	+	+
8	−	−	−	+

As just mentioned, combinations 2, 3, and 6 cannot occur. Behnke and Stein have given examples showing domains in which each of the other five combinations actually do occur. Thus five of the eight conceivable combinations occur for $n > 1$ while for $n = 1$ the first combination occurs for all (*schlicht*) domains.

We shall close this section by remarking that Stein [1] and Behnke and Stein [2] have investigated these problems on Riemann surfaces while Oka [3] and Stein [2] have obtained topological criteria for the solvability of Cousin's and Poincaré's problems for domains which are exact existence domains of some analytic function. The theory of existence domains (regularity domains) and regularity envelopes is closely related to the three problems, as various writers have observed.

5. Certain results related to Poincaré's and Cousin's problems. Various other important results related to the three problems of the previous section have been obtained. Two of the most interesting of these are due respectively to Bergman and to Bochner.

We shall first present an early result of Bergman. In presenting this result we shall follow the presentation of Behnke and Thullen [1]. In 1932 Bergman [1] first asked the following question: *given a set of functions* g_m, $m=1, 2, \cdots$, *analytic in a domain* D, *what conditions must be imposed upon the functions in order that there shall exist a function* f, *analytic in* D, *which has these functions* g_m *exactly as its zero-functions*? (A function g analytic in a domain D is called a *zero-function* of an analytic function F in D in case F vanishes at all points of D in which $g=0$.) Bergman proved the following result for E_4:

THEOREM. *Denote by* $A(g)$ *the lower limit of the volume integral*

$$\int_D |1 - v(w, z)g(w, z)|^2 dV, \qquad (dV = dxdydudv),$$

where $v(w, z)$ *runs over the totality of all non-vanishing functions analytic in* D. *If then* $\{g_m(w, z)\}$, $m=1, 2, \cdots$, *is an infinite sequence of functions analytic and of integrable square in* D *and if furthermore there exists a* $p>0$ *so that*

$$\sum_{m=1}^{\infty} [A(g_m)]^p$$

converges, then there exists a function $f(w, z)$ *analytic in* D *and having exactly the* g_m *as its zero-functions.*

Bergman has followed up this theorem with a series of interesting results on existence of analytic functions with prescribed zeros, and also similar results for zeros and singularities of meromorphic functions or harmonic functions. Some of these results are given in Bergman [2, 3, 4]. Certain applications are given in Bergman and Martin [1].

We now pass to a very brief discussion of the next result. As stated in the previous section, Poincaré proved his assertion valid for the entire space; he applied methods of potential theory to obtain this result. In extending Poincaré's work to the more general case of cylindrical domains, Cousin used the method of iterating Cauchy's integral in one complex variable. Recently Bochner [2] used a simplified approach to Poincaré's original method and connected Poincaré's theorem to Hartogs' theorem on analytic continuation from the boundary. By making use of Green's formula for general harmonic functions, he not only connected the two results mentioned but he gave new proofs of them and also obtained various new results in very general versions. One of the new results is an extension of Poincaré's theorem to functions on the torus.

6. Concluding remarks. In a talk of this nature it is necessary to limit very drastically the number of topics which can be discussed. In no way do I want

to imply that the results mentioned here are the most important or even the most typical. If there were sufficient time I would like to discuss the theory of analytic mappings, the problem of carrying Runge's theorem over to several variables, the problem of the determination of an analytic function from its boundary values on characteristic surfaces, the theory of regularity domains and regularity envelopes and the related work on various types of convexity, Levi's condition on the boundary of a domain which is the existence domain of an analytic function (a regularity domain), Fourier transforms, Hurwitz's theorem on rational functions, and many other topics. There are many important unsolved problems in the theory. Discussion of these as well as discussion of many of the topics just mentioned will be found in the excellent books of Bergman [5], Osgood [3, 4], and Behnke and Thullen [1].

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METHODS OF PRESENTING e AND π*

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1. Introduction. Initiating a student into calculus is about what sailing through the straits of Messina used to be: On one side the Charybdis dragging the boat into her whirlpool, on the other side the Scylla waiting for the vessel to shatter on her rock. The whirlpool engulfing so many teachers consists of the false statements concerning infinitely small quantities, the rock on which the beginner goes to pieces is the solid foundation of analysis. The proper initiation into calculus must painstakingly avoid all senseless statements and at the same time avoid unduly rigorous reasoning. If we add that even a first introduction should aim at conveying to the student the understanding of calculus rather than a mere mechanical ability to handle formulae, then we have about described the difficulties confronting the teacher.

The solution is to present only statements and arguments which the student can easily visualize and which are capable of rigorous proofs, but to present them without any attempt at rigorously proving them beyond what may come up in answering the questions of intelligent students. As an example of a presentation in this spirit, in what follows I outline methods of introducing e and π which for years I have found useful in teaching.

2. Concerning e. We start by plotting the exponential curves E_a given by the equations $y=a^x$ for particular values of $a(>1)$, especially for the bases $a=2$ and $a=4$. Then we compare the curve E_a with an auxiliary line L given by the equation $y=x+1$. Clearly, each of the curves E_a has the point (0, 1) in common with L. The curve $E_2(y=2^x)$ has a second point with positive abscissa in common with L, *viz.*, the point (1, 2), the curve E_4 a second point with negative abscissa, *viz.*, the point $(-1/2, 1/2)$.

The student easily realizes that if we let a increase beyond 2, the abscissa of the second point of intersection of E_a and L comes closer to 0 from the right side, and that if we let a decrease below 4, the abscissa of this point comes closer to 0 from the left. In fact, if we preassign the abscissa $\xi(>-1)$ of the second point of intersection, it is easy to find exactly one value of a for which the curve E_a intersects L at the point $(\xi, \xi+1)$. We have to satisfy the equation $a^\xi=\xi+1$, which yields $a=(1+\xi)^{1/\xi}$. The student actually computes the values of a corresponding to some small positive and negative values of ξ (for very small values of ξ by means of decadic or natural logarithmic tables), *e.g.*, for $\xi=1/2, 1/10, \cdots; -1/4, -1/10, \cdots$. On the basis of this experience he will readily believe that as ξ approaches 0, the value of a approaches a number whose first five decimals have been computed as 2.71828. He is told that this number has been given the name e while the curve E_e is called the natural exponential curve, and e^x the natural exponential function.

The student will further admit:

* An address to the teachers of calculus in the V-12 program at the University of Notre Dame.

that for $a>e$ the curve E_a intersects L at a second point, with a negative abscissa, which is the closer to 0 the closer a is to e; that each secant joining the point (0, 1) to a close neighboring point on E_a is appreciably steeper than L, and that consequently the tangent to E_a at (0, 1) has a slope >1;

that for $a<e$ the curve E_a intersects L at a second point, with a positive abscissa, which is the closer to 0 the closer a is to e; that each secant joining the point (0, 1) to a close neighboring point on E_a is appreciably flatter than L, and that consequently the tangent to E_a at (0, 1) has a slope <1;

that for $a=e$ the curve E_e with the equation $y=e^x$ does not intersect L in any second point, that the secants joining the point (0, 1) to neighboring points on E_e come the closer to L the closer the neighboring point is to (0, 1), and that consequently L is the tangent to E_e at (0, 1); in other words, that the tangent to E_e at (0, 1) has the slope 1.

Thus, in addition to introducing e, our method has made plausible this last statement, which we shall reformulate after the introduction of the concept of the derivative, by saying that the derivative of the function e^x at $x=0$ is $=1$, a fact of fundamental importance for the entire calculus. If a rigorous proof of the fact that $(e^h-1)/h$ approaches 1 as h approaches 0 is desired, it can easily be supplemented for the case that h assumes the values 1, 1/2, 1/3, $\cdots$.

Clearly, from $a=b+(a-b)$ and $a>b>1$ it follows that $a^m>b^m+mb^{m-1}(b-a)>b^m+m(b-a)$. Thus we obtain the following

LEMMA

$$|\alpha-\beta|>m|\alpha^{1/m}-\beta^{1/m}| \quad \text{if} \quad \alpha>\beta>1.$$

In order to show that with increasing m the numbers $(e^{1/m}-1)/(1/m)$ approach 1 we form the auxiliary numbers $e_m=(1+1/m)^m$ for which $e_m^{1/m}=1+1/m$. By our lemma,

$$|e^{1/m}-e_m^{1/m}|<|e-e_m|/m,$$

that is,

$$|e^{1/m}-1-1/m|<|e-e_m|/m.$$

Dividing this inequality by $1/m$ we obtain

$$\left|\frac{e^{1/m}-1}{1/m}-1\right|<|e-e_m|,$$

from which our contention follows since, by the definition of e, the numbers e_m approach e, and thus the numbers $|e-e_m|$ approach 0.

The reason for the importance of the fact that $(e^h-1)/h$ approaches 1, is that in conjunction with the functional equation $e^{x+y}=e^x\cdot e^y$ this fact enables us to differentiate e^x:

$$\frac{e^{x+h}-e^x}{h}=e^x\cdot\frac{e^h-1}{h}\to e^x \cdot \text{ as } \quad h\to 0;$$

and that from the formula $(e^x)' = e^x$ the entire differential calculus of elementary functions, excluding the trigonometric functions, can be derived by virtue of the two rules concerning the differentiation of a sum of two functions, and the differentiation of a function of a function. (For a detailed development of this "Algebra of Derivation" the reader is referred to the author's booklet *Algebra of Analysis*, Notre Dame Mathematical Lectures No. 3, 1944.)

3. Concerning π. In introducing the beginner into the differentiation of the trigonometric functions, the teacher will, of course, use the concept of π familiar to the student from elementary geometry, and the trigonometric functions as they are known from trigonometry, including the concept of radian measure. Again the essential fact which he has to learn, is that for $x=0$ the derivative of the sine, or still better of the tangent, is equal to 1. In view of the fact that $\tan 0=0$, this amounts to the statement that $\tan h/h$ approaches 1 as h approaches 0.

Again, the reason for the importance of the fact that $\tan h/h$ approaches 1, is that in conjunction with the functional equation,

$$\tan (x+y) = \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y},$$

it enables us to differentiate $\tan x$:

$$\frac{\tan (x+h) - \tan x}{h} = \frac{1}{h}\left[\frac{\tan x + \tan h}{1 - \tan h \cdot \tan x} - \tan x\right]$$

$$= \frac{1}{1 - \tan h \cdot \tan x} \cdot \frac{\tan h}{h} (1 + \tan^2 x) \to (1 + \tan^2 x) = \sec^2 x;$$

and that by means of the formula $(\tan x)' = \sec^2 x$ all the trigonometric and arc functions can be differentiated. Thus to complete the differentiation of all elementary functions we merely have to use the formulae

$$\sin x = \frac{\tan (x/2)}{1 + \tan^2 (x/2)} \quad \text{and} \quad \cos x = \frac{1 - \tan^2 (x/2)}{1 + \tan^2 (x/2)},$$

which obviously are important also for other reasons, and

$$\tan (\arctan x) = x, \qquad \sin (\arcsin x) = x.$$

(Again for details, the reader is referred to the author's *Algebra of Analysis*.)

In order to prove that $\tan h/h \to 1$ as $h \to 0$, the following idea is useful. It applies to the case that h assumes the values $\pi/3, \pi/4, \cdots, \pi/n, \cdots$ but can easily be adapted to the general case, which, in the opinion of the author, should not be presented to the beginner. If we circumscribe to the circle of radius 1 a regular polygon of n sides, then from elementary trigonometry it is clear that each side has the length $2 \tan (\pi/n)$ where π is the radian measure of two right

angles. Hence the length of the polygon is $2n \tan(\pi/n)$. As n gets larger, the lengths of the polygons approach that of the circle (elementary geometry!), that is, 2π. Dividing the formula $2n \tan(\pi/n) \to 2\pi$ by 2π, we obtain

$$\frac{\tan(\pi/n)}{\pi/n} \to 1.$$

The same reasoning applied to inscribed polygons would, of course, yield

$$\frac{\sin(\pi/n)}{\pi/n} \to 1.$$

While this procedure is quite satisfactory as a first introduction, the teacher should realize that the use of radian measure in developing the differentiation of the trigonometric functions is objectionable. The radian measure is based on the concept of the length of a circle which is the limit of the lengths of inscribed and circumscribed polygons. Whenever we consider $\tan x$ where x is measured in radians, we really presuppose a process of the same logical order as the formation of e^x. Now we certainly would not start the exposition of exponential functions with the study of e^x. We start with 2^x, 4^x, *etc.*, then proceed to the idea of a one-parameter family of functions a^x. In fact, we begin with rational values of x and then, on the basis of considerations of continuity, extend a^x to all values of x. Subsequently a procedure described in the first part of this paper singles out the natural exponential curve E_e from the curves E_a (and e from the numbers a) in such a way that for $x=0$ the derivative of the natural exponential function is $=1$.

4. Analogous methods of approach. It may thus be of interest that in a completely analogous way we may start with a one-parameter family of trigonometric functions T_p, among which those corresponding with rational values, like 2 and 4, can be as easily handled as 2^x and 4^x. Then a procedure can be developed which is completely analogous to that which we developed for exponential functions and which singles out a "natural" trigonometric function T_π from the T_p (and π from the numbers p) in such a way that for $x=0$ the derivative of the natural trigonometric function is $=1$.

We shall describe this development for the tangential functions. We call tangential curve of period p, and denote by T_p, the curve with the equation

$$y = \tan(2Rx/p).$$

The ordinate of T_p at the abscissa x is the trigonometric tangent of an angle which is $2x/p$ times a right angle. Clearly, this definition is independent of the concept of length. In order to construct T_p for a given number p, we declare that on the X-axis the point with the abscissa $p/2$ will symbolize the right angle, and the points with other abscissae will stand for proportional multiples of the right angle, *e.g.*, the point $p/4$ for one-half of a right angle, $p/6$ for one-third of a right angle, the point x for $2x/p$ times a right angle. As ordinates at these

abscissae we lay off the tangents of $R/2$, $R/3$, $2Rx/p$, respectively. For instance, we can construct T_4 exactly as we constructed E_4. We start with rational abscissae and then extend the curve T_4 by virtue of continuity considerations to all values of x.

Next we might compare the curves T_p with the auxiliary line L_0 given by the equation $y=x$. Clearly, each curve T_p has the origin in common with L_0. The curve T_4 intersects L_0 in another point with an abscissa between 0 and $p/2$ (and only this interval will be considered in what follows). The curve T_3 has only the origin in common with L_0. One readily sees that if we let p decrease below 4, the abscissa of the second point of intersection of T_p and L_0 approaches 0. In fact, if we preassign the abscissa ξ of the second point of intersection, it is easy to compute by means of trigonometric tables (in degrees or in radians) a value of p for which T_p and L_0 intersect at the abscissa ξ. We have to satisfy the equation $\tan(2R\xi/p)=\xi$, which yields

$$p = \frac{2R\xi}{\arctan \xi},$$

so that T_p has the equation

$$y = \tan \frac{\arctan \xi}{\xi} x.$$

Unfortunately, while in the theory of exponential functions the expression of the base a in terms of ξ leads to the ordinary definition of e when ξ approaches 0, for tangential functions the expression of the period p in terms of ξ does not give us a clear insight into the nature of the limit which p approaches as $\xi \to 0$. We thus shall define as "natural" tangential curve the curve T_p which has L_0 as tangent at the origin. This procedure has the double advantage of guaranteeing the existence and unicity of the natural tangent, and of implying that the natural tangential curve is T_π where π denotes the length of a semicircle of radius 1.

In fact, since every curve T_p can be obtained from any one of them by a proper change of the unit on the X-axis, thus every tangential function $t(x)$ can be obtained from any one of them, say, from $t_0(x)$, by a transformation $t(x)=t_0(cx)$ for a proper constant c, it is clear that the tangents to the various lines T_p at the origin form the pencil of straight lines through this point with the exception of the two coördinate axes; in other words, that each finite slope $\neq 0$ is associated with exactly one number p. There is exactly one curve, let us call it T_q, with the slope 1 at 0. It is easily seen that for $p<q$ the line T_p has no point other than the origin in common with L_p while in this case the sine curve S_p with the equation

$$y = \sin \frac{2R}{p} x$$

does intersect L_0 in another point, and that for $p>q$ the line T_p has another point in common with L_0.

In order to prove that $q=\pi$, we repeat our former reasoning about the regular polygons with n sides in slightly changed form. We remark that with increasing n the lengths $2n \tan (2R/n)$ approach the length of the circle. If we call this latter length 2π, then

$$\frac{\tan (2R/n)}{\pi/n} \to 1 \quad \text{as} \quad n \to \infty.$$

It clearly follows that

$$\frac{\tan \dfrac{\pi}{\pi} \dfrac{2R}{n}}{\pi/n} \to 1 \quad \text{as} \quad n \to \infty.$$

Now only for $p=q$ do we have

$$\frac{(\tan 2Rx/p)}{x} \to 1 \quad \text{as} \quad x \to 0.$$

Setting $x=\pi/n$ we see: only for $p=q$ do we have

$$\frac{\tan \dfrac{\pi}{p} \dfrac{2R}{n}}{\pi/n} \to 1 \quad \text{as} \quad n \to \infty.$$

We have seen that the above formula does hold for $p=\pi$. Hence $q=\pi$.

A PROBLEM OF REGIONS

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If a pie is cut twice, each slice intersecting the other *once* within the pie, four pieces are formed. A third cut, not passing through the point of intersection of the first two, intersects the existing slices in two points (0-cells) which form on the third cut three 1-cells, the partial boundaries of the three new partitions of the pie. Hence three cuts yield $4+3=7$ pieces, one of which is *interior*, *i.e.* does not touch the boundary or crust of the pie.*

This note is concerned with a generalization of this problem. In d-space, let $S(n, d, k)$ be the maximum number of d-cells formed by the partitioning of a simply connected euclidean region R_d by n hyperplanes of $d-1$ dimensional subspaces, each hyperplane intersecting each other in k $d-2$-cells within R_d, and no $d+1$ or more hyperplanes passing through a $d-2$-cell; and let $C(n, d, k)$

* The author, this MONTHLY, vol. 33, pp. 466–469. Recent work on allied subjects will be found in this MONTHLY as follows: W. B. Carver, The polygonal regions into which a plane is divided by n straight lines, vol. 48, pp. 667–675; R. C. Buck, Partition of space, vol. 50, pp. 541–544; Solution to problem E554, vol. 50, pp. 564–565.

be the number of interior regions formed. For the example given above, $S(3, 2, 1) = 7$ and $C(3, 2, 1) = 1$. Suppose we consider any plane region R_2 and $k = 3$. The cutting elements are isomorphic to cubical parabolas; each will intersect the other three times within R_2 and will cut the boundary twice.

By reasoning similar to that used in the first paragraph, in general, the number of new regions created by an nth cut are the same as the number of partitions $S(n-1, d-1, k)$, hence

$$S(n, d, k) = S(n-1, d, k) + S(n-1, d-1, k),$$

and a similar equation for $C(n, d, k)$. Integrating the difference equations and adjusting the constants, we have

$$S(n, d, k) = \sum_{i=0}^{d} \binom{n}{i} + (k-1)\binom{n}{d},$$

and

$$C(n, d, k) = \binom{n-1}{d} + (k-1)\binom{n}{d} \quad \text{where} \quad n \geqq 2,\ n \geqq d,$$

and

$$C(n, n, k) = k - 1.$$

Consider the partitions of a plane by curves isomorphic to circles, *i.e.* each curve intersects each other in two points ($k = 2$), and no circle intersects the boundary R_2. If the circles are opened into straight lines, the number of partitions lost are the same as the number of interior regions $C(n, 2, 1)$. Let prime letters be used in this case. Hence

$$S'(n, 2, 2) = S(n, 2, 1) + c(n, 2, 1).$$

For the extension of this case, which occurs only when k is even, we obtain

$$S'(n, d, k) = \sum_{i=0}^{d} \binom{n}{i} + \binom{n-1}{d} + (k-2)\binom{n}{d},$$

and

$$C'(n, d, k) = \binom{n-1}{d-2} + k\binom{n}{d}, \quad \text{where} \quad n > 0,\ d > 1,$$

and

$$S'(0, d, k) = C'(0, 1, k) = 1, \qquad C'(n, 1, k) = nk.$$

An interesting result occurs when $n > d$,

$$S(n, d, k) = S'(n, d, k) = 2^n.$$

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

FEUERBACH'S THEOREM BY "MEAN POSITION"

HOWARD EVES, Syracuse University

The "Theory of Mean Position" often furnishes an elegant method for finding distances between notable points, and in this capacity may be successfully employed in proving the famous Feuerbach Theorem (that the nine-point circle of a triangle is tangent to the inscribed and the three escribed circles). Although I devised the following proof several years ago it is still, so far as I am aware, new, and might therefore be worth adding to the host of already existing solutions of the Feuerbach Theorem.

FUNDAMENTAL THEOREM. *If O is the mean center of a system of points $A, \cdots$ for the system of multiples $a, \cdots$, and if P is any point, then*

$$\sum (a\overline{AP}^2) = \sum (a\overline{AO}^2) + \sum (a)\overline{OP}^2. \tag{1}$$

This is a well known and basic theorem in the "Theory of Mean Position." See, *e.g.*, art. 55 in M'Clelland's *The Geometry of the Circle*.

LEMMA. *If O is the incenter of triangle ABC with opposite sides, a, b, c, P any point, and $2s=a+b+c$, then*

$$\sum (a\overline{AP}^2) = abc + 2s\overline{OP}^2. \tag{2}$$

The incenter of a triangle is the mean center of the vertices for multiples proportional to the opposite sides. If, then, in (1) we let P coincide with A, B, C in succession we get

$$\begin{aligned} bc^2 + cb^2 &= \sum (a\overline{AO}^2) + \sum (a)\overline{AO}^2, \\ ca^2 + ac^2 &= \sum (a\overline{AO}^2) + \sum (a)\overline{BO}^2, \\ ab^2 + ba^2 &= \sum (a\overline{AO}^2) + \sum (a)\overline{CO}^2. \end{aligned}$$

Multiplying the first by a, the second by b, the third by c, and adding, we get

$$2abc(a+b+c) = (a+b+c)\sum (a\overline{AO}^2) + (a+b+c)\sum (a\overline{AO}^2),$$

whence

$$\sum (a\overline{AO}^2) = abc.$$

Hence the relation (1), where A, B, C are the vertices of a triangle, a, b, c the opposite sides, O the incenter, and P any point, becomes the relation (2).

FEUERBACH'S THEOREM. *The nine-point circle of a triangle is tangent to the inscribed and the three escribed circles.*

Let N be the center of the nine-point circle. Then, for $P \equiv N$, (2) becomes

$$\sum (a\overline{AN}^2) = abc + 2s\overline{ON}^2. \tag{3}$$

Now, if R is the circumradius,

$$\overline{AN}^2 + \overline{BN}^2 = \tfrac{1}{2}c^2 + \tfrac{1}{2}R^2, \tag{4}$$

$$\overline{CN}^2 + \overline{AN}^2 = \tfrac{1}{2}b^2 + \tfrac{1}{2}R^2, \tag{5}$$

$$\overline{BN}^2 + \overline{CN}^2 = \tfrac{1}{2}a^2 + \tfrac{1}{2}R^2. \tag{6}$$

(The sum of the squares of two sides of a triangle is equal to one half the square of the third side plus twice the square of the median on that side.)

Adding (4) and (5) and subtracting (6), and then multiplying by $\frac{1}{2}a$, we find

$$a\overline{AN}^2 = \tfrac{1}{4}a(R^2 + b^2 + c^2 - a^2).$$

Similarly

$$b\overline{BN}^2 = \tfrac{1}{4}b(R^2 + c^2 + a^2 - b^2),$$

$$c\overline{CN}^2 = \tfrac{1}{4}c(R^2 + a^2 + b^2 - c^2).$$

Therefore

$$\begin{aligned}\sum (a\overline{AN}^2) &= \tfrac{1}{4}R^2(a + b + c) + \tfrac{1}{4}(ab^2 + ac^2 + bc^2 + ba^2 + ca^2 \\ &\quad + cb^2 - a^3 - b^3 - c^3) \\ &= \tfrac{1}{2}R^2s + \tfrac{1}{4}[8(s - a)(s - b)(s - c) + 2abc] \\ &= \tfrac{1}{2}R^2s + \tfrac{1}{4}[8r\Delta + 8R\Delta],\end{aligned}$$

where Δ is the area of the triangle and r is the inradius. Substituting the above in (3) we find

$$\tfrac{1}{2}R^2s + 2r\Delta + 2R\Delta = 4R\Delta + 2s\overline{ON}^2,$$

or

$$\overline{ON}^2 = \tfrac{1}{4}R^2 - Rr + r^2 = (\tfrac{1}{2}R - r)^2.$$

That is, the difference between the radii of the nine-point circle and the incircle equals the distance between the centers of these circles, and consequently the circles touch internally.

Similarly, if O_a is the excenter opposite A, we may show that

$$-a\overline{AN}^2 + b\overline{BN}^2 + c\overline{CN}^2 = -abc + 2(s - a)\overline{O_aN}^2,$$

a relation corresponding to (3). Proceeding as above we finally find

$$O_aN = \tfrac{1}{2}R + r_a,$$

whence the nine-point circle and the escribed circle opposite A touch externally.

Hence the theorem.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent to American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the editors or officers of the Association.

Exact Values of the first 200 Factorials. By H. S. Uhler. New Haven, H. S. Uhler, 1944. 4+18 pages. $0.80.

This booklet consists of an introduction and a table of the factorials of the integers from 1 to 200. Spacing is used to separate groups of ten digits in expressing the factorials of numbers greater than 13. Thus 24! appears as 6204 4840173323 9434360000. Use is made of the device of writing the number of terminal zeros (enclosed in parentheses) at the right of the other digits though the number of digits so indicated is always a multiple of 5. Thus 25! appears as 1 5511210043 3309859840 (5)—a number with six terminal zeros. The table gives exact values of the first 200 factorials and a value of the reciprocal of 155! with an error of less than one unit in the 578th decimal place.

In the brief introduction the author discusses his purpose in computing these tables, the methods of computation and checking, and the consistency with previously published results. It appears that the results here published are new for $120 < n \leqq 200$ and that considerable diligence has been exercised in seeing that the published results are free from error.

P. S. DWYER

Mathematical and Physical Principles of Engineering Analysis. By W. C. Johnson. New York, McGraw-Hill Book Company, Inc., 1944. 10+346 pages. $3.00.

This book is intended as a text for engineering students during an upperclass year and presents the mathematical methods appropriate to the solution of problems in various branches of engineering, together with the physical background. The treatment is eminently practical and the explanations are clear, concise and interesting. It represents an original piece of work, possible only by an author who has had much experience in solving engineering problems and who possesses the skill to pass on his experience to his pupils. Mathematical rigor would be out of place in such a book and is disclaimed.

To analyze an engineering problem, it is explained in the first chapter, one must first determine what the problem is, what data are needed, what methods of attack are available and which is the most promising, and what simplifying assumptions should be made. Then the physical conditions are put in mathematical form, the solution is carried out analytically and numerically, being checked by appropriate methods, and finally the results are summarized in useful form.

Chapter 2 reviews "some basic physical principles," properly emphasizing the concept of energy and relating mainly to mechanics and electricity. (The following statement on p. 23 requires modification: "It has been found experimentally that a change in flux linkage of 10^8 maxwell-turns per second will induce an emf of 1 volt." This is really a matter of definition, both as to the general physical relation and as to the units—although, of course, there are experimental facts behind it.)

Linear differential equations and their initial conditions are briefly introduced in Chapter 3 on "transient and steady-state conditions" and are more fully discussed in Chapter 4 on "setting-up equations." The "graphical and numerical solution of differential equations" in the next chapter first takes up quadratures and then step-by-step solutions, the particular methods emphasized being commendable. The long Chapter 6 on "ordinary differential equations" discusses analytical solutions, the greater part being devoted to linear equations with constant coefficients. Although operational notation and ideas are used freely, complete operational solutions unfortunately are not given. Such operational solutions are easier and can be presented more briefly than the methods of the text. Moreover all cases can be embraced in a single procedure; and only a single evaluating formula is required. At the end of this chapter appears a discussion of Bessel's equation and Bessel functions, together with the gamma function.

Chapter 7 takes up "vector representation of sinusoids," through complex numbers, as long used for alternating-current circuits, and applies it also to mechanical vibrations. "The checking of equations" in the next chapter includes dimensional checking and checking by taking simple limiting cases. (On p. 207 appears the statement that "nearly all physical equations are dimensionally homogeneous"; it would be more satisfying if the word "nearly" were omitted and there were added the clause, *if all dimensional constants are written.*)

An unusually clear presentation of "dimensional analysis" in Chapter 9 includes the changing of units and the derivation of general formulas. (The matter of "dimensional constants" is rather briefly dismissed at the end; it might better have been brought out that the method of dimensional analysis really fails in certain cases, in the sense that the dimensions of such a constant are not known until the desired law has been formulated: thus a student would be disappointed if he tried to find the adiabatic or the isothermal $p-v$ law of an ideal gas.)

Fourier series are treated in Chapter 10, with numerical methods of harmonic analysis and a brief introduction to the Fourier integral. (The third example of a "periodic function" on p. 239 is, of course, non-periodic—as is explained later, on p. 249.) The final chapter, on "systems with distributed constants," introduces partial differential equations. Various problems are worked out in terms of the elementary functions, either in closed form or as Fourier series. Graphical field plotting is applied to two-dimensional problems and, more briefly, to three-dimensional problems having an axis of symmetry.

Besides the problems worked out in the text, there are given a large number

of practice problems, some formal and many of a practical nature. It is to be regretted that no answers are given.

The question arises, by whom can the subject matter of this book best be taught? A teacher having an engineering point of view is called for; and he will most often be selected from an engineering department. But a mathematics department would be the stronger if it possessed such a man. In any case, the book will be of great value to teachers of mathematics (as well as to teachers of physics), to show the purest of them "how the other half lives."

ALAN HAZELTINE

Sistemas de Ecuaciones Analiticas en Terminos Finitos, Diferenciales y en Derivadas Parciales. (Monografias publicadas por la Facultad de Ciencias Matematicas, Fisico-Quimicas y Naturales, applicadas a la Industria, Universidad Nacional del Litoral, No. 1.) By Beppo Levi. Rosario, Argentina, 1944. 218 pages. $8.00 m/n.

This is volume one of a series of monographs to be published in Spanish by the Faculty of Mathematical, Physical-chemical and Natural Sciences of the National Coastal University (Universidad Nacional del Litoral), Rosario, Argentina. According to the fly-leaf, volumes two, three and four have also been published, but titles are not given; presumably these do not deal with mathematical subjects. The author of the present volume is director of the Mathematical Institute of the above-mentioned university.

As announced by the title, Professor Levi has presented a theory of analytic equations in several complex variables, embracing finite, ordinary differential and partial differential equations. (Reviewer's note: For the sake of English sentence structure we have substituted the phrase "finite equation" in place of "equation in finite terms.") To paraphrase the opening words of the introduction, "The problems related to the resolution of equations, whether finite or differential, assume fundamentally different aspects according to whether we postulate only the existence of such properties of continuity and differentiability of the given functions and the unknowns as are strictly necessary if a particular problem is to have meaning, or admit additional hypotheses relative to the existence of a certain number of successive derivatives, or finally grant at once the existence of all derivatives." Although it seems clear that the author is most intrigued by the first of these three alternatives, he evidently feels that the last has greatest importance for physical applications, and promises that it will be adopted in the monograph. Nevertheless there are to be found some twenty-five notes in small type scattered throughout the book, some of which will be of aid to a student wishing to relax the hypotheses. In addition there are several pages of bibliographical notes at the end of the volume.

The monograph is not to be regarded as a textbook in the American sense; it contains a number of examples but no problems. Chapter I begins with some brief but illuminating remarks on complex numbers and passes on to the study

of complex power series in several complex variables. By means of analytic continuation the functions studied are defined throughout their natural domain of existence. Derivatives are introduced directly from the power series, and an analogue is developed of the Cauchy Integral Theorem. The problem of solving a holomorphic equation $f(x_1, x_2, \cdots, x_n) = 0$ for one of the variables is attacked in a more or less algebraic fashion via the Weierstrass Preparation Theorem and leads to the notion of an algebroid function. Little is said however about the difficult problem of classifying types of singularity of an analytic function of several variables. Chapter II introduces the Jacobian in connection with the solution of finite equations, and also considers the solution of systems of first-order ordinary differential equations.

In Chapter III the author turns aside from the main train of thought in order to prove two elementary algebraic lemmas, useful for the sequel. As evidence of the careful scholarship which has gone into this book we should like to quote (in paraphrase) the second of these: "Lemma II. Suppose given a finite system M of monomials in k variables $x_1, x_2, \cdots, x_k$, such that no monomial of the system is a multiple of any other. We shall show that *there can be determined (often in various ways) another system N of monomials, and, at the same time, certain groups of variables* (multiplier variables) *in such a manner that all monomials which are not multiples of any monomial of M may be obtained in one and only one way by multiplication of a monomial of N by a monomial formed from the corresponding multiplier variables.*"

Chapter IV gives an extended treatment of finite systems of analytic partial differential equations of finite order in a finite number of variables. The notion of a complete system is fundamental. An orderly method is presented whereby a given system may either be shown to be "impossible" or be reduced to an equivalent system comprising a finite number of complete systems. Most of the examples of the book appear in this chapter, and there are many references to the literature. The latter part of the chapter is devoted to the study of initial conditions and to the formal calculation of series solutions. Chapter V offers a necessary complement to the preceding with convergence proofs for the solutions of certain normal systems. In the notation of the author: "*A system of partial differential equations always has an analytic solution satisfying given initial conditions analytic with respect to a normal ordering of the derivatives.*" There are also given certain extensions of this result. The tract concludes with a few remarks on Cauchy conditions and characteristic varieties.

The reviewer did not make a careful analysis of the proofs throughout the whole of the book; however a detailed examination of Chapter I sufficed to give him complete confidence in the author. There appear to be few misprints and only one minor misstatement was noted. In conclusion we should like to remark upon the elegance of mathematical expression which is possible in Spanish and to suggest that with the present volume the new series of monographs has made a most satisfactory beginning.

R. H. Bruck

Spherical Trigonometry. By Aaron Freilich, Henry Shanholt, and Joseph Seidlin. New York, Silver Burdett Co., 1943. 4+140 pages. $1.28.

Here is a text-book which is designed to meet the needs of students who have had a course in plane trigonometry and who need some knowledge of spherical trigonometry and its application to problems in navigation. Inasmuch as many, if not the majority, of such students will not have the background of solid geometry, it is necessary to include some sort of an introduction to those theorems which form the foundation for the development of spherical trigonometry.

As the authors note, this introduction has been made on a rather informal, intuitive basis, omitting, with a casual "it can be proved" proofs which could have been supplied with a few additional lines. The inclusion of such proofs, while not seriously increasing the size of the book, would render its study more intellectually satisfying.

The development of the formulas for solution of spherical triangles is essentially the same as that to be found in any standard text on the subject. Exercises for students of varying ability and needs are provided with completely worked illustrative examples, which should help the student in developing an adequate problem solving technique.

The usefulness of the text might have been improved by providing haversine tables, since the haversine formulas are introduced. Incidentally this might bring the work more in line with current navigation practice.

The conversational style adopted in much of the treatment has its drawbacks, since many of the conclusions drawn are dependent upon the student's producing the correct answers to the questions posed. In actual classroom practice, the teacher would have to see that the correct answers are produced, together with the reasoning which leads to the stated conclusion. These gaps in the reasoning would detract materially from the value of the book for self instruction.

The presence of occasional historical notes should stimulate the interest of the student. Cumulative reviews offer an excellent means for testing progress in acquiring the desired knowledge.

In the hands of a capable teacher, the text should prove to be very useful.

H. P. PETTIT

NEW BOOKS RECEIVED

Methods of Advanced Calculus. By Philip Franklin. New York and London, McGraw-Hill Book Co., Inc., 1944. 12+486 pages. $4.50.

Introduction to the Theory of Divergent Series. By Otto Szász. (Lectures by Otto Szász, written by Joshua Barlaz.) Cincinnati, University of Cincinnati, 1944. 6+72 pages, lithoprinted. $1.25.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 651. *Proposed by E. D. Schell, Arlington, Virginia*

You have eight similar coins and a beam balance. At most one coin is counterfeit and hence underweight. How can you determine whether there is an underweight coin, and if so, which one, using the balance only twice?

E 652. *Proposed by V. Thébault, San Sebastián, Spain*

In which scales of notation can a four-digit number $aabc$ be the square of a two-digit number mn, if $c=b+1$ and $n=m+1$?

E 653. *Proposed by J. H. Butchart, Grinnell College*

The ends of a chord UV of the circle $r=a$ have the parametric angles ϕ and $k\phi$, where k is a constant greater than 1. Show that the locus of the midpoint of UV is a prolate epitrochoid whose polar equation is

$$r = a \cos \frac{k-1}{k+1}\theta.$$

Show also that the envelope of UV is an epicycloid, the point of contact dividing UV in the ratio $1:k$.

E 654. *Proposed by D. H. Browne, Buffalo, N. Y.*

Let $n!_1$ denote the coefficient of $x^n/n!$ in the expansion of $e^x/(1-x)$. Show that, for a prime p and any nonnegative integer k,

$$\sum_{n=k}^{k+p-1} n!_1 \equiv -1 \pmod{p}.$$

(*Cf.* E 488 [1942, 478].)

E 655. *Proposed by W. C. Rufus, Observatory of the University of Michigan*

The first term of a geometrical progression is a three-digit number, and the ratio is 9. Each term is divisible by the sum of its digits. Find the sequence having the largest number of terms. (The one beginning with 100 has thirteen terms.)

SOLUTIONS

A Class of Convergent Series

E 615 [1944, 162]. *Proposed by H. S. Wall, Northwestern University*

Let $g_1, g_2, g_3, \cdots$ be real numbers such that $0<g_p<1$ and such that the series

$$g_1 + (1 - g_1)g_2 + (1 - g_2)g_3 + \cdots$$

converges to a sum not exceeding unity. Establish the convergence of

$$1 + \sum_{p=1}^{\infty} \frac{g_1 g_2 \cdots g_p}{(1 - g_1)(1 - g_2) \cdots (1 - g_p)}.$$

Solution by M. F. Smiley, U. S. Naval Academy. Rewrite the sum of the first p terms of the first series as $S_p = T_p + g_p$, where

$$T_p = g_1(1 - g_2) + g_2(1 - g_3) + \cdots + g_{p-1}(1 - g_p).$$

By hypothesis $\{T_p\}$ is strictly increasing and bounded by unity. Consequently this sequence has a limit $t>0$. Setting $s = \lim S_p$, we have

$$g = \lim g_p = s - t \leqq 1 - t < 1.$$

Since the limit of the pth term of the first series is zero, we deduce that $g=0$. The ratio test then ensures the convergence of the second series.

Also solved by Murray Barbour, Henry Helson, W. J. Thron, and the proposer. Thron followed the proposer in seeing a connection with continued fractions and citing Perron, *Die Lehre von den Kettenbrüchen* (second edition), p. 259.

A Variant of Gregory's Series for π

E 618 [1944, 231]. *Proposed by Thorold Gasset, Cambridge, England*

Prove that, for any value of θ between 0 and π,

$$\sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots = \frac{\pi}{4}.$$

Solution by Irving Kaplansky, New York, N. Y. This is a standard result in Fourier series (see, *e.g.*, Goursat, *Mathematical Analysis*, vol. 1, p. 421). However, the following formal derivation (in the spirit of eighteenth century mathematics) may be of interest. Let

$$f(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \tfrac{1}{2}\log\frac{1+x}{1-x}.$$

Then

$$f(e^{i\theta}) = \tfrac{1}{2}\log\frac{1+e^{i\theta}}{1-e^{i\theta}} = \tfrac{1}{2}\log(i\cot\tfrac{1}{2}\theta)$$

$$= \begin{cases} \tfrac{1}{2}\log\cot\tfrac{1}{2}\theta + i\pi/4 & (0<\theta<\pi), \\ \tfrac{1}{2}\log(-\cot\tfrac{1}{2}\theta) - i\pi/4 & (\pi<\theta<2\pi). \end{cases}$$

Hence

$$\sin\theta+\frac{\sin 3\theta}{3}+\frac{\sin 5\theta}{5}+\cdots=\begin{cases}\pi/4 & (0<\theta<\pi),\\ -\pi/4 & (\pi<\theta<2\pi),\end{cases}$$

and also

$$\cos\theta+\frac{\cos 3\theta}{3}+\frac{\cos 5\theta}{5}+\cdots=\tfrac{1}{2}\log\left|\cot\frac{\theta}{2}\right|.$$

Also solved by D. H. Browne, Howard Eves, Edward Fleisher, Bernard Greenspan, R. W. Hamming, H. D. Lipsich, E. B. Roessler, and P. D. Thomas. The following literature was cited: Bromwich, *Theory of Infinite Series*, p. 356; Jackson, *Fourier Series and Orthogonal Polynomials* (Carus Monograph No. 6), p. 209; Knopp, *Theory and Application of Infinite Series* (1928), p. 374; Loney, *Plane Trigonometry* (1925), p. 122; Woods, *Advanced Calculus*, p. 298.

The Quadrangle of Circumcenters

E 619 [1944, 231]. *Proposed by W. B. Clarke, San José*

Prove that the four triangles of the complete quadrangle formed by the circumcenters of the four triangles of any complete quadrilateral are similar to those triangles.

Solution by Howard Eves, Syracuse University. Let $a_1a_2a_3a_4$ be the complete quadrilateral and $A_1A_2A_3A_4$ the complete quadrangle, so that A_1 is the circumcenter of triangle $a_2a_3a_4$, and so on. Now (see art. 196 in Johnson's *Modern Geometry*) the circumcircles (A_1), (A_2), (A_3), (A_4) are concurrent at a point P. Let c_{34} and c_{41} denote the lines joining P to the points of intersection (a_3, a_4) and (a_4, a_1). Since these lines are perpendicular respectively to A_1A_2 and A_2A_3, we have

$$\sphericalangle c_{34}c_{41}=\sphericalangle A_1A_2A_3.$$

But $\sphericalangle c_{34}c_{41}=\sphericalangle a_3a_1$, these being inscribed in the same arc of circle (A_2). Hence $\sphericalangle A_1A_2A_3=\sphericalangle a_3a_1$. Similarly, for all suitable i, j, k, we have

$$\sphericalangle A_iA_jA_k=\sphericalangle a_ka_i.$$

This proves the theorem.

Triangles with a Given Ratio for Two Angles

E 620 [1944, 231]. *Proposed by Alan Wayne, Rhodes School, New York*

Find integral sides for a triangle in which one angle is six times another.

Solution by R. C. Buck, Cambridge, Mass. Consider a triangle ABC with angle $B=nA$ and side $a=1$. We will find all such triangles with *rational* sides. (Then, multiplying by the common denominator of b and c, we shall have all integral solutions for one angle n times another.) Applying the law of sines to ABC (with $a=1$), we obtain

$$b=R_n,\qquad c=R_{n+1},\tag{1}$$

where $R_n = (\sin nA)/(\sin A)$. In terms of $r = 2\cos A$, a well known formula gives

$$(2) \qquad R_{n+1} = r^n - \binom{n-1}{1} r^{n-2} + \binom{n-2}{2} r^{n-4} - \binom{n-3}{3} r^{n-6} + \cdots;$$

e.g.,

$$R_6 = r(r^2 - 3)(r^2 - 1), \qquad R_7 = r^2(r^2 - 3)(r^2 - 2) - 1.$$

If r is rational, so is R_k, and hence so are b and c; conversely, if b and c are rational, so is r, for $r = (b^2+c^2-1)/bc$. In order that a value of r may give a valid triangle, we must have

$$0 < A + B < \pi;$$

in terms of r, this means that

$$(3) \qquad 2\cos\frac{\pi}{n+1} < r < 2.$$

This completes the general solution. For any rational r satisfying (3), we can find b and c by (1) and (2), and these will be rational. Conversely, every rational solution is given by some rational r.

For the case when $n = 6$, we choose any rational p/q such that

$$1.80194 \leqq p/q < 2.$$

Then multiplication by q^6 yields a triangle with integral sides

$$a = q^6,$$
$$b = pq(p^2 - 3q^2)(p^2 - q^2),$$
$$c = p^2(p^2 - 3q^2)(p^2 - 2q^2) - q^6.$$

The smallest solution is given by $p/q = 11/6$:

$$a = 46656, \qquad b = 72930, \qquad c = 30421.$$

Here are the solutions for smaller values of n.

$n = 2$: $a = q^2$, $b = pq$, $c = p^2 - q^2$; *e.g.* (4, 6, 5).

$n = 3$: $a = q^3$, $b = q(p^2 - q^2)$, $c = p(p^2 - 2q^2)$; *e.g.* (8, 10, 3).

$n = 4$: $a = q^4$, $b = pq(p^2 - 2q^2)$, $c = p^4 - 3p^2q^2 + q^4$; *e.g.* (81, 105, 31).

$n = 5$: $a = q^5$, $b = q(p^4 - 3p^2q^2 + q^4)$, $c = p(p^2 - 3q^2)(p^2 - q^2)$; *e.g.* (1024, 1220, 231).

Also solved by Murray Barbour, H. N. Carleton, R. W. Hamming, Frank Hawthorne, E. P. Starke, and the proposer. The proposer points out that, if (a_n, b_n, c_n) is the general primitive solution with $B = nA$, the expressions can be quickly written down by means of the recurrence formulas

$$a_{n+1} = qa_n, \qquad b_{n+1} = qc_n, \qquad c_{n+1} = pc_n - qb_n.$$

In particular, we always have $a_n = q^n$.

The proposer remarks, further, that repeated application of the identity

$$\sin (n+1)A = r \sin nA - \sin (n-1)A,$$

where $r = 2 \cos A$, yields the expansion

$$\frac{b_n}{c_n} = \frac{R_n}{R_{n+1}} = \frac{\sin nA}{\sin (n+1)A} = \frac{1}{r} - \frac{1}{r} - \cdots \frac{1}{r}$$

(to n components). Setting $r = p/q$ (or $\cos A = p/2q$), we deduce that b_n/c_n is the nth convergent of the continued fraction

$$\frac{q}{p} - \frac{q^2}{p} - \frac{q^2}{p} - \cdots;$$

e.g., with $p/q = 11/6$,

$$\frac{b_6}{c_6} = \frac{6}{11} - \frac{36}{11} - \frac{36}{11} - \frac{36}{11} - \frac{36}{11} - \frac{36}{11} = \frac{72930}{30421}.$$

A Periodic Continued Fraction

E 621 [1944, 285]. *Proposed by C. D. Olds, Purdue University*

Show directly (*i.e.*, without substituting θ into the value given by the standard algebraic proofs) that

$$\frac{1}{a} + \frac{1}{a} + \cdots = e^{-\theta},$$

where a is a positive number and $\theta = \text{arg sinh } (a/2)$.

I. *Solution by E. D. Schell, Arlington, Virginia.* Assuming the continued fraction to be convergent, there is no loss of generality in calling its value $e^{-\theta}$. Then

$$e^{\theta} = a + \frac{1}{a} + \cdots = a + e^{-\theta},$$

$$2 \sinh \theta = a,$$

and

$$\theta = \text{arg sinh } \tfrac{1}{2}a.$$

II. *Solution by the proposer.* In terms of $\theta = \text{arg sinh } \frac{1}{2}a$, the successive convergents are

$$\frac{\cosh \theta}{\sinh 2\theta}, \quad \frac{\sinh 2\theta}{\cosh 3\theta}, \quad \frac{\cosh 3\theta}{\sinh 4\theta}, \quad \frac{\sinh 4\theta}{\cosh 5\theta}, \cdots.$$

The limit of this sequence is clearly $e^{-\theta}$.

Also solved by Murray Barbour.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4145. *Proposed by Howard Eves, Syracuse University*

Find the positions of three non-overlapping circles in a triangle which have a maximum combined area.

4146. *Proposed by G. B. Lang, Emory University*

Suppose that n men check their hats, and then each man takes a hat at random. Let $g(n)$ denote the probability that no one gets his own hat. Find the value of $g(n)$ and its limit for $n \to \infty$.

4147. *Proposed by V. Thébault, San Sebastián, Spain*

Two circles varying in magnitude and position roll on two fixed circles. Find the loci of their centers of similitude. If the straight line of their centers has a constant direction, the midpoint of the segment of their centers describes a straight line.

SOLUTIONS

The Game of Fourteen

4093 [1943, 516]. *Proposed by B. M. Stewart, Michigan State College*

A store conducted the following game in which the player paid 10¢, and if he won received \$1 in merchandise. Out of five throws, each time throwing ten dice, a win was declared if the player had *fourteen* or more appearances of the numbered face named by him before playing; the player had the additional advantage that on the *first* throw of ten dice he could count the face occurring the greatest number of times as if it were the special face he had selected. The problem is to compare the theoretical probability with the odds offered by the store.

Solution by A. G. Clark, Colorado State College. The only complicating factor is the privilege of choosing the face which occurs the greatest number of times on the first throw. If $i = 2, 3, \cdots, 10$ denotes the greatest number of times any face whatever occurs on the first throw, then the probability of such occurrence is given by

$$\sum_i \frac{6!10!6^{-10}}{\sum_{j=1}^{i} (j!)^{n_j} n_j!} \quad \text{or} \quad \sum_i \text{ for brevity,}$$

where the n_j are non-negative integers such that

$$\sum_{j=1}^{i} jn_j = 10$$

and the summation for a given value of i is over all combinations of the n_j which satisfy this condition.

The probability that i occurrences of some face will be followed by a total of $14-i$ or more occurrences of that face on the next four throws of the ten dice will evidently be

$$6^{-40} \sum_{r=14-i}^{40} \binom{40}{r} 5^{40-r}$$

which for purposes of computation can be expressed in terms of beta functions, *i.e.*,

$$I_{1/6}(14 - i, 27 + i) = 1 - I_{5/6}(27 + i, 14 - i).$$

The probability of winning the game will then be

$$\sum_{i=2}^{10} I_{1/6}(14 - i, 27 + i) \sum_i .$$

A tabular arrangement of the computation follows. Interpolations for $I_{1/6}(14-i, 27+i)$ use only first differences.

i	$I_{1/6}(14-i, 27+i)$	$\sum_i$	Product
2	.026	.0675	.00176
3	.058	.5293	.03070
4	.116	.3105	.03602
5	.211	.0781	.01648
6	.347	.0130	.00451
7	.509	.0015	.00076
8	.677	.0001	.00007
9	—	—	—
10	—	—	—
	The probability of winning is		.09030.

Thus, the player's expectation is about 9¢.

Note: A more interesting and much more commonly played variant of this game is for the player to let lie those dice which turn up the face he has chosen and to recast the remaining dice each time. The player would win if at the conclusion of five casts the total number of favorable dice amounts to say seven or more.

Solved also by E. P. Starke and the proposer. The other two solutions are similar except in the modifications of the final formula for computation. Starke's result is .09108 and the proposer's result lies between .091066 and .091067.

An Alternant Type of Determinant

4098 [1943, 569]. *Proposed by P. R Halmos, Urbana, Illinois*

Evaluate the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & & x_n^2 \\ x_1^4 & x_2^4 & x_3^4 & & x_n^4 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ x_1^{2^{n-1}} & x_2^{2^{n-1}} & x_3^{2^{n-1}} & & x_n^{2^{n-1}} \\ x_1 & x_2 & x_3 & \cdots & x_n \end{vmatrix}.$$

Solution by Howard Eves, Syracuse University In 1841 Jacobi established the following factorization of the general simple alternant:

$$(1) \quad \left| x_1^{a_1}, x_1^{a_2}, \cdots, x_1^{a_n} \right| = \zeta^{1/2}(x_1x_2 \cdots x_n) \left| H_{a_1-n+1}, H_{a_1-n+2}, \cdots, H_{a_1} \right|,$$

where $\zeta^{1/2}(x_1x_2 \cdots x_n)$ is Sylvester's notation for the continued product

$$(x_1 - x_2)(x_1 - x_3) \cdots (x_1 - x_n)(x_2 - x_3) \cdots (x_2 - x_n) \cdots (x_{n-1} - x_n)$$

and where H_j is the complete homogeneous polynomial with unit coefficients of degree j in $x_1, x_2, \cdots, x_n$.

Using this theorem the determinant of the problem can be factored as

$$\left| x_1, x_1^2, x_1^4, \cdots, x_1^{2^{n-1}} \right| = (x_1x_2 \cdots x_n)\zeta^{1/2}(x_1x_2 \cdots x_n)D,$$

where

$$D = \begin{vmatrix} H_{-n+1} & H_{-n+2} & \cdot H_{-3}H_{-2}H_{-1}H_0 \\ H_{-n+2} & H_{-n+3} & \cdot H_{-2}H_{-1}H_0\ H_1 \\ H_{-n+4} & H_{-n+5} & \cdot H_0\ H_1\ H_2\ H_3 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ H_{-n+2^{n-1}} & H_{-n+2^{n-1}+1} & \cdot \cdots H_{2^{n-1}-1} \end{vmatrix}.$$

Since $H_{-j}=0$ and $H_0=1$, D reduces to

$$(2) \quad -\begin{vmatrix} H_{-n+4} & H_{-n+5} & \cdot H_0 & H_1 \\ \cdot & \cdot & \cdot & \cdot \\ H_{-n+2^{n-1}} & H_{-n+2^{n-1}+1} & \cdot H_{2^{n-1}-4} & H_{2^{n-1}-3} \end{vmatrix}.$$

Denoting the determinant of the problem by Δ_n, and the determinant in (2) by H, we then have

$$\Delta_n = -(x_1x_2 \cdots x_n)\zeta^{1/2}(x_1x_2 \cdots x_n)H,$$

where H is a determinant of order $n-2$. It seems difficult to improve much on this partial expansion of Δ_n.

For the special cases $n=3$ and $n=4$ we find

$$\Delta_3 = -x_1x_2x_3(x_1-x_2)(x_1-x_3)(x_2-x_3)(x_1+x_2+x_3)$$

and

$$\Delta_4 = -x_1x_2x_3x_4(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_4)(H_5-H_1H_4).$$

Jacobi's relation (1) was established in an improved manner by N. Trudi in 1864. Others who have published the relation are Clebsch and Gordan (1868), Naegelsbach (1871), Malet (1874), Kostka (1875), Cayley (in a note in Salmon's *Modern Higher Algebra*, 3rd ed. 1876), Garbieri (1878), and Mansion (1879). See vol. iii, topic *Alternants*, of Muir's *History of Determinants*, or Chapter XI of *Theory of Determinants* by Muir and Metzler.

Pedal Curves

4099 [1943, 569]. *Proposed by J. H. Butchart, Grinnell College*

If P is any point of a curve and Q is the corresponding point of the pedal with respect to the point O, then OQ makes the same angle with the pedal that OP makes with the curve.

Solution by A. Sisk, Maryville, Tenn. Suppose the given curve c is well behaved, with simple tangents near P. Let P' be near P on c, and suppose the tangents to c at P and P' intersect in T. Take Q' as the image of P'. Let angles OTQ' and OQQ' be denoted by θ and ϕ. Since the points O, T, Q, Q' are cyclic, $\theta=\phi$. Hence the limiting values of these angles are equal as P' approaches P along c, causing T to approach P along PQ and Q' to approach Q along the pedal.

For this problem and others kin to it, see Hilton's *Plane Algebraic Curves*, p. 166 · · · ·.

Solved also by H. Demir, H. Eves, S. Hughart, H. Siller, and P. D. Thomas.

Editorial Note. The solution by Eves is similar to the above; he gave the above reference and also Williamson's *Differential Calculus*, 1927, p. 227, and remarked that this is an old and well known result usually established in that manner. Siller remarked that the problem is solved in the second reference. Hughart uses polar coordinates and derivatives. Thomas uses rectangular coordinates and derivatives.

Demir proved the theorem for the case where the angle PQO is an arbitrarily given angle by considering the instantaneous center I of rotation of the rigid system of two distinct straight lines QP and QO, the first tangent at P to the curve $\{P\}$ and the second passing through the fixed point O. For any given position of P the point I is the point of concurrence of the normals PI, QI to the curves $\{P\}$, $\{Q\}$, and the perpendicular to QO at O. If QT and PQ are corresponding directions of the tangents for this position, then it easily follows that angles OQT, OPQ, QIO are equal. For the theory of the point I he referred to Mannheim, *Principes et développements de géométrie cinématique*. However, this theory is quite simple in this case and probably is given in many books on kinematics and some on geometry.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

At New York University Associate Professor F. W. John has been promoted to a professorship and H. E. Wahlert to an assistant professorship.

In the department of mathematics and mechanics of the University of Minnesota, Assistant Professors H. A. Doeringsfeld and F. E. Miller have been promoted to associate professorships; Assistant Professors H. L. Turrittin and N. R. Amundson have been granted leaves of absence, the former to serve as a mathematician with the Operations Analysis Division of the Army Air Force and the latter as Fellow in the Applied Mechanics Program at Brown University.

Associate Professor L. E. Babcock of Newberry College, South Carolina, has been appointed to an assistant professorship at the University of Richmond.

Ralph Mansfield of Chicago Teachers' College has been appointed research engineer with Jos. Weidenhoff, Inc., Chicago, Illinois.

Helen K. Milleson of Carleton College has been appointed statistician in the Bureau of Census, Washington, D. C.

Assistant Professor Edwin Nilson of Mount Holyoke College has been given leave of absence to serve in the United States Navy.

Professor C. F. Thomas of Case School of Applied Science has been appointed head of the mathematics department.

Assistant Professor C. J. Thorne of Louisiana State University has been granted leave of absence to serve as development engineer in the Curtiss Wright Corporation at Bloomfield, New Jersey.

The following appointments to instructorships are announced:
Coe College (Iowa): Aletha C. Gaddis
Sunflower Junior College, Moorhead, Mississippi: Virginia Felder
University of Buffalo: R. C. Luippold
University of New Mexico: E. Marie Hove

Professor J. S. Miller of Emory and Henry College, Emory, Virginia, died March, 16, 1944.

Ruth Newlin of Creston Junior College and High School, Creston, Iowa, died October 20, 1944.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

AID TO LIBRARIES IN WAR AREAS

Up to the end of 1943, $160,000 had been spent for subscriptions to 325 scholarly and scientific journals, to be stored in this country, for distribution after the war to libraries in war areas. The money is provided by a grant from the Rockefeller Foundation, which has allotted from $50,000 to $70,000 annually for this purpose since 1941. The fund is administered by the Committee on Aid to Libraries in War Areas of the American Library Association, and inquiries should be addressed to Miss Edith A. Wright, Committee on Aid to Libraries in War Areas, Library of Congress Annex, Study 251, Washington, D. C.

Originally the committee was formed to study ways in which libraries in this country could help similar institutions abroad in their struggle with wartime conditions, and especially to find some means of preserving for later distribution U. S. publications not available because of the war. The Rockefeller grant has enabled the committee to implement its findings.

The American Library Association and the constituent societies of the American Council of Learned Societies have prepared lists of outstanding publications in their various fields of interest which were published in this country during 1939–44. They will be available to individuals and groups interested in the selection of books for foreign libraries.

In the case of periodicals, several factors influence orders placed for each title, namely, the journal's foreign circulation before the war, the ratio of foreign circulation to total circulation, the number of free or exchange subscriptions, the number of copies available through gifts from subscribers in this country, and the relative importance of the journal to research. In no case have purchases equaled more than half the number of paid institutional subscriptions in Europe and Asia discontinued because of the war. For this reason, and because no funds are available for purchasing periodicals from individuals nor for the acquisition of volumes issued before 1939, the committee has been gratified to note the efforts of governments-in-exile and other groups to maintain periodical subscriptions for important libraries in various countries.

The committee's periodical purchases are supplemented by gifts. In this way current issues and back files of important periodicals have been secured for use in restocking libraries that have been damaged or destroyed. Free storage space is difficult to obtain, however, and whenever possible donors are urged to store materials until central storage space is available. Transportation costs are paid,

but material accepted for storage is limited to what seems of first importance. Gifts are being stored in a number of libraries in all parts of the country, and offers of additional storage space are always welcome. Prospective donors are asked to first report titles and dates of the journals available to the office of the committee. Shipping instructions will then be issued, indicating where and how shipment should be made. The committee has prepared three lists of desirable periodicals, which are available on request: a general list, a list of medical journals, and a list of technical periodicals.

No definite decisions as to disposal of books and periodicals can be made until more is known of library conditions in war areas. Present efforts of the committee are therefore confined to collecting material, obtaining all available information about the needs of libraries, and recording what is being done in the same field by other organizations.

The mathematical journals especially desired as gifts are the following: American Journal of Mathematics, American Mathematical Monthly, Bulletin of the American Mathematical Society, Transactions of the American Mathematical Society, Journal of the American Statistical Association, Annals of Mathematics, Annals of Mathematical Statistics, Duke Mathematical Journal, Journal of Mathematics and Physics, Mathematical Reviews, Quarterly of Applied Mathematics, Review of Economic Statistics, School Science and Mathematics, and Scripta Mathematica. The committee is now purchasing some copies of all these journals with the funds made available by the Rockefeller Foundation. However, it is certain that additional copies will be needed. There is a serious demand, also, for long runs of the various journals published before the war to replace those that have been destroyed. In addition to the titles listed in mathematics, there is a need for similar journals in other fields and some of the general scientific magazines, such as Science and The Scientific Monthly.

EDUCATION OF STUDENTS FROM WAR-TORN COUNTRIES

A conference on providing educational opportunities for students from the war-torn countries was held in Philadelphia, February 28–29, 1944; the consultation included representatives of twenty-four national educational associations and representatives of six agencies of the Federal Government. The Institute of International Education and the Association of American Colleges joined with The Edward W. Hazen Foundation in sponsoring the conference. A general statement of principles adopted by the group follows.

"It is evident that the war-torn countries will be desperately in need of assistance in educational and cultural reconstruction after the war. Our people very properly will wish the Government of the United States to cooperate in this program. In doing so, the first step may be to negotiate bilateral agreements for the reception and exchange of students and teachers; and to do this in collaboration with the institutions and agencies representing both publicly supported and privately supported education in the United States. We believe that education of students from other countries is primarily the function of our existing

educational institutions. We anticipate, however, that the magnitude of the task will require supplemental support from our Government, in addition to that which will be supplied by other Governments, the institutions themselves, and by other agencies. In cooperation with the U. S. Office of Education, other Government departments, and with educational institutions, and other agencies, the Department of State should formulate the general policies governing this program, based upon the determined needs. It should also negotiate agreements with foreign governments and obtain, in collaboration with other agencies, the necessary data on which the program is to be founded. It should survey existing and proposed legislation and immigration procedures, with the view to facilitating the entrance and residence of students from other countries by removing unnecessary restrictions upon such students and also with the view of extending privileges enjoyed by such students to trainees from other countries who come for training in industrial, business or government agencies. The educational institutions and agencies should provide information as to their resources and facilities. They should undertake to adapt their regular curricula to the students coming to the United States or organize other programs to meet the particular needs. Where possible, they should provide some assistance to the students. The sponsors of this conference are authorized to organize a small provisional continuing committee, consisting of representatives of educational associations of the country. This continuation committee shall cooperate with officials appointed by the Government and will assist in formulating policies regarding an appropriate program. It should plan procedures for enlisting the resources of educational institutions and agencies making use of available experience and facilities. It should enlist public interest and support."

The continuation committee of the conference has further investigated the problems involved. This led to recognition of a need for some cooperative agency in the United States to give continuing leadership in the field. A proposal for such an agency has been drafted. This confidential document has been submitted to the Division of Cultural Cooperation of the Department of State for study and appropriate action. It is to be expected that when a decision regarding next steps has been reached, colleges, universities, and educational associations will be informed promptly. In the meantime, information concerning educational facilities that may be available for students from war-torn countries is being assembled.

ACADEMIC STANDARDS FOR V-12 OFFICER CANDIDATES

Previous to the start of the Navy V-12 term upon November 1, 1944, commanding officers of the various units were requested to scrutinize the academic qualifications of men completing their V-12 studies upon that date. The exact instruction follows.

"Any student who is not in good standing academically upon completion of his allowed number of terms in the Navy V-12 Program, regardless of the fact that he may have successfully completed the *minimum* course requirements for the type of candidacy to which he has been assigned, shall not be considered

academically qualified for further training as an officer candidate, nor shall a student who fails to obtain a satisfactory grade with respect to officer aptitude be continued in the program. A mark of less than 2.5 (on the Navy 4.0 basis) is regarded as unsatisfactory."

Presumably, the requirements explained in this communication will be maintained for the duration of the V-12 Program. A mark of 2.5 is approximately equivalent to an average grade of D upon the usual scale A, B, C, D, and F.

CIVIL SERVICE APPOINTMENTS FOR MATHEMATICIANS

On October 9, 1944, the U. S. Civil Service Commission announced that applications would be accepted for war service appointment as "mathematician." Although many mathematicians have been employed in war research by agencies of the Federal Government, there has been limited recognition of mathematics as a profession until recent months. Thus, the announcement of the Civil Service Commission is of considerable interest.

The duties of the mathematician are defined as follows:

A. To plan, direct, conduct or assist in the planning or conducting of mathematical research.

B. To make specialized computations.

C. To make investigations in the field of applied mathematics involving (1) the analyses of practical and theoretical problems requiring the use of advanced mathematical theory and principles and (2) the interpretation of mathematical results in terms of physical concepts so that the results may be used in engineering practice and scientific work.

D. To develop and introduce new mathematical procedures and check current mathematical practices.

The qualifications of the mathematician are described below.

Grade P-1. At least 3 years of progressive technical experience in mathematics requiring an intimate knowledge of the theory and the applications of the principles of all the following: algebra, trigonometry, analytical geometry, and differential and integral calculus. This experience must be of such scope and character as to demonstrate clearly that the applicant's training and ability are adequate for the conduct of the mathematical duties indicated above. Salary: $2,000+overtime pay of $433.

In addition to the minimum requirement specified for the P-1 grade, applicants for grades P-2 and above must show additional experience of a progressively higher level as follows:

Grade P-2. At least 2 years of professional work in mathematics, involving the use of the principles of theoretical or applied mathematics in the solution of scientific and related problems. Salary: $2,600+overtime pay of $563.

Grade P-3. At least 3 years of progressive professional experience in mathematics of such character as to demonstrate capacity for original mathematical research or the ability to plan and develop mathematical projects and to prepare full reports of work accomplished. Salary: $3,200+overtime pay of $628.

Grade P-4. At least 5 years of responsible progressive professional experience in mathematics of such character as to demonstrate marked capacity for original research, outstanding professional attainments, or the ability to plan, administer, or perform productive research in mathematics and to develop mathematical projects. Credit will be given for all valuable experience of the type required, regardless of whether compensation was received or whether the experience was gained in a part time or full time occupation. Such experience will be credited on the basis of time actually spent in appropriate activities. Salary: $3,800+overtime pay of $628.

Nonqualifying Experience. Experience involving routine computations or statistical compilations, work in pure statistics, or high school teaching of mathematics will not be accepted as qualifying professional experience in mathematics.

Substitution of Education for Experience. Successful completion of pertinent undergraduate study in a college or university of recognized standing may be substituted for the required experience on the basis of 1 year of academic study for each 9 months of experience, provided that for each year of academic study substituted, the applicant must show at least 5 semester hours in algebra, trigonometry, analytical geometry, differential and integral calculus, the solution of differential equations, or other courses in higher mathematics.

Graduate study in mathematics successfully completed in a college or university of recognized standing may be substituted year for year for the experience prescribed for these positions up to a maximum of 3 years of experience. In order to substitute the full 3 years of graduate study, applicants must have completed all the requirements for the Ph.D. degree, including the thesis.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE MAY MEETING OF THE WISCONSIN SECTION

The twelfth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Milwaukee Downer College on Saturday, May 13, 1944. Sessions were held in the morning and in the afternoon, with Professor May M. Beenken, Chairman of the Section, presiding.

There were twenty-six in attendance, including the following thirteen members of the Association: R. H. Bardell, Ethelwynn R. Beckwith, May M. Beenken, W. W. Bigelow, R. C. Huffer, Lionel London, C. C. MacDuffee, Sister Mary Felice, A. C. Moeller, R. E. Norris, H. P. Pettit, J. I. Vass, Louise A. Wolf.

At the business meeting the following officers were elected for the coming year: Chairman, Morris Marden, University of Wisconsin at Milwaukee; Program Committee, Sister Mary Felice, Mount Mary College, B. R. Ullsvik, Eau Claire State Teachers College. It was voted that the next meeting be held in

May, 1945, at Milwaukee State Teachers College, the exact date to be set by that institution.

The following papers were presented:

1. *The Hagge circle of a point in the plane of a triangle*, by Sister Mary Felice, Mount Mary College.

This paper was a summary of an article entitled *Der Fuhrmannsche Kreis und der Brocardsche Kreis als Sonderfälle eines allgemeineren Kreises* which was published by K. Hagge in 1836. The Hagge circle of the incenter of a triangle is the Furmann circle of that triangle, while that of the centroid is the Brocard circle of the triangle formed by the lines drawn through the vertices of the triangle parallel to the opposite sides. The Hagge circle of a point on the circumference of the circumcircle of the triangle provides a proof for Steiner's theorem on the collinearity of the orthocenters of the triangles of a complete quadrilateral. Finally, proofs by the methods of pure geometry were given for some theorems enunciated by A. M. Peiser in his paper *The Hagge circle of a triangle* which was published in this MONTHLY, vol. 49, 1942, pp. 524–527. Peiser had proved the theorems by the method of conjugate coördinates.

2. *Groups, quasigroups, and Cayley squares*, by Dr. R. H. Bruck, University of Wisconsin, introduced by Professor Kenney.

Dr. Bruck discussed the interrelations of the theories of groups, loops, quasigroups, and Cayley, or latin, squares. He outlined some of the fundamental theorems common to these subjects, and touched upon certain applications to algebra, geometry, and biometry. He pointed out that Cayley squares may be used to give concrete meaning to abstract concepts such as those of homomorphism and isotopy.

3. *Determination of latitude in an emergency*, by Joel Brenner, Lawrence College, introduced by Professor Marden.

Mr. Brenner's paper was published in this MONTHLY, vol. 51, 1944, p. 343.

4. *The rise of matrices and normal coördinates for the solution of torsional vibration problems*, by K. E. Bisshopp, Fairbanks-Morse Company, introduced by Mr. Battig.

Matrix methods are convenient for obtaining numerical solutions of problems involving the calculation of normal modes and frequencies of torsional vibration in systems with many degrees of freedom. In general, the types of systems considered have no fixed points, so that it is necessary to remove the rigid body mode of vibration corresponding to the degenerate frequency $p=0$ before inverting the stiffness matrix. The contribution of this paper is the presentation of a general method for computing the inverse matrix of a free system comprising an arbitrary number of masses connected by elastic shafts without mass. The usual iterative process based upon Sylvester's theorem then can be used to evaluate the frequencies successively in increasing order of magnitude

with the aid of the orthogonality relations between the normal modes. The elements of the modal columns obtained by iteration are proportional to the factors required for the reduction of the original system to normal coördinates. The practical advantage of the normal coördinate method is that a partial fraction expansion for the forced vibration amplitudes can be constructed directly from the matrix results, thereby avoiding a tedious torque summation calculation.

5. *Functions and services of the Armed Forces Institute*, by Lt. Col. C. W. Hansen, AGD, Commandant, United States Armed Forces Institute, introduced by Miss Wolf.

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P. L. TRUMP, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Eighth Summer Meeting, Montreal, Canada, June 23–25, 1945.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA, Cedar Rapids, April 14, 1945
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D. C., May, 1945
METROPOLITAN NEW YORK, Brooklyn, April 21, 1945
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA
OHIO, Columbus, April 5, 1945
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles March 10, 1945
SOUTHWESTERN
TEXAS
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WISCONSIN, Milwaukee, May, 1945

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